Approximate Solution of Differential Equations of Fractional Orders Using Bernstein-Bézier Polynomial

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Abstract

In this paper the Bernstein-Bézier polynomial curves are used to approximate the solution of the initial value problems involving derivatives of fractional order IVPDFO in the sense of Riemann-Liouville definition.
Some examples are given to show the efficiency and simplicity of the our method.

Introduction

Differential equations involving derivatives of fractional order have shown to be adequate models for various physical phenomena in areas like damping, diffusion processes, etc.
Several approximate methods have been proposed to find approximate solution of differential equations of fractional order. These methods include quadrature formula approach ( Diethelm, 1997 ) [1], numerical scheme for a fractionally damped model ( Yuan & Agrawal, 1998) [2], PECE ( Predict, Evaluate, Correct, Evaluate ) type approach ( Diethelm and Freed, 1999) [3], Adams-type predictor-corrector method ( El-Sayed, El-Mesiry and El-Saka, 2004) [4], and least squares finite-element (Fix and Roop, 2004) [5].
In this paper, we have developed a simple method to solve IVPDFO using Bernstein-Bézier polynomial curves.

The Initial Value Problem (IVP)

Let \( \alpha \in ( n , n+1 ] , \alpha_k \in ( k-1 , k ] , k = 1, 2, \ldots, n \) and \( \alpha_0 = 0 \)
,consider the initial value problem [4],

\[
D^\alpha x(t) = f( t, x(t), D^{\alpha_1} x(t - r), D^{\alpha_2} x(t - 2r), \ldots, D^{\alpha_n} x(t - nr) ) , t \in I
\]

\[
D^j x(t)=0 \quad \text{for} \quad t \leq 0 , \quad j = 0, 1, \ldots, n
\]
In this paper, we discuss approximate solution to the multi-term fractional orders differential equation
\[ D^k x(t) = f(t, x(t), D^{a_1} x(t), D^{a_2} x(t), \ldots, D^{a_m} x(t)), \quad t > 0 \quad \ldots(1) \]
subject to the initial values
\[ D^j x(0) = c_j, \quad j = 0, 1, \ldots, k-1 \quad \ldots(2) \]
where \( \alpha_i, i = 1, 2, \ldots, m \) are real numbers such that
\[ 0 < \alpha_1 < \alpha_2 < \ldots < \alpha_m < k \]
and \( k \) is any positive integer number.

Here, \( D^\alpha \) denotes the Riemann-Liouville fractional derivative of order \( \alpha, \ n-1 < \alpha \leq n \) and \( n \) is any integer number of the function \( x(t) \), defined by [6,7]
\[ D^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-u)^{n-\alpha-1} x(u) \, du \]
or
\[ D^\alpha x(t) = \frac{1}{\Gamma(-\alpha)} \int_0^t (t-u)^{-\alpha-1} x(u) \, du \quad \ldots(3) \]
where \( \Gamma \) is the gamma function.

**BERNSTEIN-BÉZIER POLYNOMIAL CURVES**

One of the most elegant proofs given by S. N. Bernstein in 1912. He introduced the following polynomials for a function defined on [0,1], [8,9].
\[ B_n f(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} f\left(\frac{i}{n}\right), \quad n=1,2,\ldots \quad \ldots(4) \]

Bernstein polynomials can be defined on an interval [a,b] by ,[10]
\[ B_{i,n}(t) = \binom{n}{i} (t-a)^i (b-t)^{n-i} \frac{(b-t)^i}{(b-a)^n} \quad i=0,1,\ldots,n \quad \ldots(5) \]
These polynomials form a partition of unity, that is
\[
\sum_{i=0}^{n} B_{i,n}(t) = 1, \quad t \in [a,b].
\]

A general Bézier curve of order \( n \), defined by \( n+1 \) vertices, can be expressed as follows [10]
\[
B(t) = \sum_{i=0}^{n} a_i B_{i,n}(t) \quad t \in [a,b]
\]
where \( a_i \) denotes the \( i \)-th vertex and give information about shape of the curve \( B \). Bézier extended the idea of the approximation of a function to the approximation of a polygon, in which \( n+1 \) vertices of a polygon are approximated via the Bernstein basis. Hence it is also called a Bernstein–Bézier polynomial curve.

To determine an approximate solution of eq.1 with eq.2, \( x(t) \) is approximated in the Bernstein polynomial basis on \( [a,b] \) as
\[
x(t) = \sum_{i=0}^{n} a_i B_{i,n}(t) \quad \ldots(6)
\]
where \( B_{i,n}(t) \) defined in eq.5 and \( a_i \) (\( i = 0, 1, \ldots, n \)) are unknown constants to be determined.

In this paper, we will use Bernstein polynomials defined on \( [0,b] \), that is
\[
B_{i,n}(t) = \binom{n}{i} \frac{t^i (b-t)^{n-i}}{b^n} \quad i=0,1,\ldots,n \quad \ldots(7)
\]
This equation can be written in terms of the power basis. This can be calculated using the binomial theorem and rewriting the summation, we have
\[
B_{i,n}(t) = \sum_{j=0}^{i} a_j (-1)^{i-j} \binom{n}{i} \binom{j}{i} \left( \frac{t}{b} \right)^i \quad i=0,1,\ldots,n \quad t \in [0,b] \quad \ldots(8)
\]

**SOME IMPORTANT PROPOSITIONS**

A general formation to fractional \( (\alpha) \) derivative and \( n^{th} \) derivative of the \( x^m \) are given throughout the following propositions:

**Proposition (1):**
The fractional \( (\alpha) \) derivative of \( x^m \), \( m \geq 0 \) is formulated as
\[ D^\alpha x^m = \frac{m! x^{m-\alpha}}{\Gamma(m - \alpha + 1)} \]

**Proposition (2):**

The \( n \)th derivative of \( x^m \) , \( m \geq 0 \) is formulated as

\[
D^n x^m = \begin{cases} 
0 & m < n \\
 m! & m = n \\
 \frac{m!}{(m-n)!} x^{m-n} & m > n 
\end{cases}
\]

These propositions can be proved by using mathematical induction. The formulas in propositions (1 and 2) is found to be very useful in our computational work.

**THE APPROXIMATION METHOD**

To determine an approximate solution of eq.1 , \( x(t) \) is approximated in the Bernstein–Bézier polynomial basis on \([0,b]\).

Substitution of eq.6 into eq.1 , we obtain

\[
D^k \sum_{i=0}^{n} a_i B_{i,n}(t) = f(t, \sum_{i=0}^{n} a_i B_{i,n}(t), D^\alpha_{i,1} \sum_{i=0}^{n} a_i B_{i,n}(t), ..., D^\alpha_{n,n} \sum_{i=0}^{n} a_i B_{i,n}(t))
\]

Then

\[
\sum_{i=0}^{n} a_i D^k B_{i,n}(t) = f(t, \sum_{i=0}^{n} a_i B_{i,n}(t), \sum_{i=0}^{n} a_i D^\alpha_{i,1} B_{i,n}(t), ..., \sum_{i=0}^{n} a_i D^\alpha_{n,n} B_{i,n}(t)) \quad (9)
\]

Let \( B_{i,n}(t) \) as in eq.8 , that is

\[
D^k B_{i,n}(t) = d_i \ D^k t^i \quad \text{and} \quad D^\alpha_{i,1} B_{i,n}(t) = d_i \ D^\alpha_{i,1} t^i
\]

where

\[
d_i = \sum_{j=0}^{i} a_j (-1)^{i-j} \binom{n}{i} \binom{i}{j} \frac{1}{b^i}
\]
and \( i=0,1,...,n \), \( r=1,...,m \)

Using propositions (1 and 2) in eq.10, we get

\[
D^k B_i,n(t) = \begin{cases} 
0 & \text{i<k} \\
\frac{i!}{d_i} & \text{i=k} \\
\frac{i!}{(i-k)!} t^{-k} d_i & \text{i>k}
\end{cases} 
\text{and } D^q B_i,n(t) = \frac{i! t^{i-\alpha_r}}{\Gamma(i-\alpha_r+1)!} d_i
\]

where \( i=0,1,...,n \), \( r=1,...,m \)

Now, applying the initial conditions eq.2 into eq.6 with eq.8 to determine the unknown constants \( a_j 's \), \( j = 0, 1, ..., k-1 \).

Then substituting eq.11 and putting \( t = t_p , \ p = 0, 1, ..., n-k \) ('s being chosen as suitable distinct points in \([0,b]\) ) in eq.9, we obtain the linear system of \((n-k+1)\) equations in the \((n-k+1)\) unknowns constants \( a_j 's \), \( j = k, ..., n \). This system can be easily solved by standard methods.

Finally, \( a_i \), \( i = 0, 1, ..., n \) are used in eq.6 with eq.8 to obtain the unknown function \( x(t) \) approximately.

**SOME EXAMPLES**

In order to check this method, we have tested it on some examples of the IVPDFO.

Example (1):

Consider the following equation [4]

\[
D^2 x(t) + D^{3/2} x(t) + x(t) = t + 1 + \frac{1}{\sqrt{\pi t}} - \frac{1}{2t\sqrt{\pi t}} \quad 0 \leq t \leq 5 \quad ...(12)
\]

Subject to

\[
x(0) = x'(0) = 1
\]

with the exact solution \( x(t) = t + 1 \)

At first, assume the approximate solution is in the form

\[
x(t) = \sum_{i=0}^{2} \sum_{j=0}^{i} a_j (-1)^{i-j} \binom{n}{i} \binom{t}{j} \left( \frac{t}{5} \right)^i
\]

\[
= a_0 + (-2a_0 + 2a_1) \frac{t}{5} + (a_0 - 2a_1 + a_2) \frac{t^2}{25}
\]

\[
= a_0 + (-2a_0 + 2a_1) \frac{t}{5} + (a_0 - 2a_1 + a_2) \frac{t^2}{25}
\]

\[
= a_0 + (-2a_0 + 2a_1) \frac{t}{5} + (a_0 - 2a_1 + a_2) \frac{t^2}{25}
\]
By using initial conditions, we obtain

\[ x(0) = a_0 = 1 \]
\[ x'(0) = \frac{1}{5} \left( -2a_0 + 2a_1 \right) = 1 \implies a_1 = \frac{7}{2} \]

Putting values of \( a_0 \) and \( a_1 \) in eq.13 to get

\[ x(t) = 1 + t + \left( -6 + a_2 \right) \frac{t^2}{25} \quad \ldots(14) \]

Now, substitution eq.14 into eq.12 and using propositions (1 and 2) we have

\[ (2 + \frac{4\sqrt{t}}{\sqrt{\pi}} + t^2) a_2 = 12 + \frac{24\sqrt{t}}{\sqrt{\pi}} + 6t^2 \]

Let \( t_0 = 0 \implies a_2 = 6 \)

That is, eq.14 becomes \( x(t) = 1 + t \)

Table(1) present a comparison between the exact and approximate solution depending on least square errors with \( h=0.1 \).

Example (2):
Consider the following equation

\[ D^2 x(t) + D^{\alpha_2} x(t) + Dx(t) + D^{\alpha_1} x(t) + x(t) = f(t) \quad \ldots(15) \]

\[ 0 < \alpha_1 < 1 \quad , \quad 1 < \alpha_2 < 2 \]

and

\[ f(t) = 2t + \frac{t^{-\alpha_2}}{\Gamma(1-\alpha_2)} + \frac{2t^{3-\alpha_2}}{\Gamma(4-\alpha_2)} + t^2 + \frac{t^{-\alpha_1}}{\Gamma(1-\alpha_1)} + \frac{2t^{3-\alpha_1}}{\Gamma(4-\alpha_1)} + 1 + \frac{1}{3} t^3 \]

Subject to \( x(0) = 1 \quad , \quad x'(0) = 0 \) with exact solution \( x(t) = 1 + \frac{1}{3} t^3 \)

Assume the approximate solution is in the form

\[ x(t) = \sum_{i=0}^{3} \sum_{j=0}^{i} a_j (-1)^{i-j} \binom{n}{i} \binom{i}{j} t^i \quad t \in [0,1] \]

\[ = a_0 + \left( -3a_0 + 3a_1 \right) t + (3a_0 - 6a_1 + 3a_2) t^2 + (-a_0 + 3a_1 - 3a_2 + a_3) t^3 \]

Using initial conditions, we obtain \( a_0 = 1 \) and \( a_1 = a_0 = 1 \)
Then $x(t)$ is become

$$x(t) = 1 + (-3 + 3a_2) t^2 + (2 - 3a_2 + a_3) t^3 \ \ \ \ \ \ ...(16)$$

Substitution eq.16 into eq.15 and using propositions (1 and 2), we have

$$\left(6 - 12t - 6t^2 - 3t^3 + \frac{6t^{2-a_2}}{\Gamma(3-a_2)} - \frac{18t^{3-a_2}}{\Gamma(4-a_2)} + \frac{6t^{2-a_1}}{\Gamma(3-a_1)} - \frac{18t^{3-a_1}}{\Gamma(4-a_1)}\right)a_2 +$$

$$\left(6t + 3t^2 + t^3 + \frac{6t^{3-a_2}}{\Gamma(4-a_2)} + \frac{6t^{3-a_1}}{\Gamma(4-a_1)}\right)a_3 = 6 - 4t - 2t^2 - \frac{5}{3}t^3 + \frac{6t^{2-a_2}}{\Gamma(3-a_2)}$$

$$- \frac{10t^{3-a_2}}{\Gamma(4-a_2)} + \frac{6t^{2-a_1}}{\Gamma(3-a_1)} - \frac{10t^{3-a_1}}{\Gamma(4-a_1)}$$

For $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{3}{2}$ with let $t_0 = 0$ and $t_1 = 1$, we obtain two equations after solve these equations we get $a_2 = 1$ and $a_3 = 1.3333$, that is

$$x(t) = 1 + 0.3333 t^3$$

the exact and approximate results are shown in table(2) depending on least square errors with h=0.1.

Example (3):
Consider the following equation

$$D^2 x(t) + D^{1/2} x(t) + x(t) = e^t \left[ 2 + \text{erf}(\sqrt{t}) \right] + \frac{1}{\sqrt{\pi t}} \ \ \ \ ...(17)$$

Subject to

$$x(0) = x'(0) = 1$$

with the exact solution $x(t) = e^t$

the exact and approximate results are shown in table(3) depending on least square errors with h=0.1.
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Amaal

Table 1: Exact and approximate results for example (1)

<table>
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<tr>
<th>t_r</th>
<th>Exact x(t)</th>
<th>Appr. result</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0.1</td>
<td>1.1</td>
<td>1.1000</td>
</tr>
<tr>
<td>0.2</td>
<td>1.2</td>
<td>1.2000</td>
</tr>
<tr>
<td>0.3</td>
<td>1.3</td>
<td>1.3000</td>
</tr>
<tr>
<td>0.4</td>
<td>1.4</td>
<td>1.4000</td>
</tr>
<tr>
<td>0.5</td>
<td>1.5</td>
<td>1.5000</td>
</tr>
<tr>
<td>0.6</td>
<td>1.6</td>
<td>1.6000</td>
</tr>
<tr>
<td>0.7</td>
<td>1.7</td>
<td>1.7000</td>
</tr>
<tr>
<td>0.8</td>
<td>1.8</td>
<td>1.8000</td>
</tr>
<tr>
<td>0.9</td>
<td>1.9</td>
<td>1.9000</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>2.0000</td>
</tr>
</tbody>
</table>

(L.S.E.) x(t) | 0.0000

Table 2: Exact and approximate results for example (2)

<table>
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<th>t_r</th>
<th>Exact x(t)</th>
<th>Appr. result</th>
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<tr>
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<td>1.3333</td>
<td>1.3333</td>
</tr>
</tbody>
</table>

(L.S.E.) x(t) | 2.1982e-009

Table 3: Exact and approximate results for example (3)

<table>
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<th>t_r</th>
<th>Exact x(t)</th>
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<td>2.7187</td>
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</table>

(L.S.E.) x(t) | 2.3721e-007
CONCLUSIONS
In our method $\alpha_1, \alpha_2, \ldots, \alpha_m$ take arbitrary values such that $0 < \alpha_1 < \ldots < \alpha_m < k$ where $k$ is any positive integer number.
Here a very simple and straight forward method, based on approximation of the unknown function of IVPDFO on the Bernstein-Bézier polynomial basis defined on some interval $[0,b]$.
Use of this method produces very accurate results. It may be mentioned that the linear systems avoid appearance of any ill-conditioned matrix.

REFERENCES