On $r$- Compact Spaces

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Abstract
The main purpose of this paper is to introduce and study a new type of compact spaces which is $r$-compact space where a topological space $(X, \tau)$ is said to be an $r$-compact space if every regular open cover of $X$ has a finite subfamily whose closures cover $X$. Several properties of an $r$-compact space are proved.

1. Introduction
Compactness occupies a very important place in topology and so do some of its weaker and stronger forms, one of these forms is H-closedness where the theory of such spaces was introduced by Alexandroff and Urysohn in 1929. In 1969, nearly compact spaces was introduced by M. K. Singal and Asha Aathur. Another type of compact space which is $S$-compact space was introduced in 1976 by Travis Thompson. From time to time several other forms of compactness have been studied. In this work, we shall study a new weaker form of compact spaces, namely $r$-compact space.

Now, let $(X, \tau)$ be a topological space, and let $A \subseteq X$, we say that:

i) $A$ is a regular open set in $X$ if and only if, $A = \overset{o}{A}$, (Dugundji, 1966).
ii) $A$ is a regular closed set in $X$ if and only if, $\overset{c}{A} = \overset{o}{A}$, (Dugundji, 1966).
iii) $A$ is a semi open set in $X$ if and only if, there is an open set $U$ such that $U \subseteq A \subseteq \overset{o}{U}$, (Levine, 1963).

Some time we use $X$ to denote the topological space $(X, \tau)$ and we will the symbol $\Box$ to indicate the end of the proof.

1.1 Remark:

i) The complement of every regular open (closed) set is a regular closed (open) set, (Dugundji, 1966).
ii) Every regular open set is an open set, (Dugundji, 1966).
iii) Every open set is a semi open set, (Levine, 1963).
iv) From (ii) and (iii), we can get that every regular open set is a semi open set.

2. Preliminaries
In this section, we introduce and recall the basic definitions needed in this work.

First, we state the following definition:

2.1 Definition (Willard, 1970): A space $X$ is said to be compact if and only if, every open cover of $X$ has a finite sub cover of $X$. 

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Now, we introduce the concept of r-compact in the following definition:

**2.2 Definition:** A space X is said to be r-compact if and only if, every regular open cover of X has a finite subfamily whose closures cover X.

Next, we recall the following definition which we needed in the following sections.

**2.3 Definition (Willard, 1970):** A space X is said to be extremely disconnected if and only if, the closure of every open set in X is also open in X.

### 3. Relationship between r-compact and some types of compact spaces

At once, we recall the definition of some types of compact spaces, as follows:

**3.1 Definition (Willard, 1970):** A space X is said to be:

1. quasi H-closed if and only if, every open cover of X has a finite subfamily whose closures cover X,(Cameron, 1978).
2. nearly-compact if and only if, every open cover of X has a finite subfamily, the interiors of the closures of which cover X,(Herrington, 1974).
3. S-closed if and only if, every semi open cover of X has a finite subfamily whose closures cover X,(Thompson, 1976).

The above notations are related in the following diagram:

It is clear that the implications 1,2,3 and 4 hold. Next, we prove 5,6,7 and 8, respectively.

**3.2 Theorem:** Every compact space is an r-compact space.

**Proof:** Let X be a compact space and let \( \{ V_\alpha \}_{\alpha \in \Omega} \) be a regular open cover of X. Since X is a compact space, So, there exist \( V_{\alpha_1}, \ldots, V_{\alpha_n} \) such that \( X = \bigcup_{i=1}^{n} V_{\alpha_i} \). Since \( V_{\alpha_i} \) is a regular open set in X, for each \( \alpha_i \in \Omega \) and for each \( i=1,2,\ldots,n \). So, \( V_{\alpha_i} = V_{\alpha_i}^\circ \) for each \( \alpha_i \in \Omega \) and for each \( i=1,2,\ldots,n \). Hence, \( X = \bigcup_{i=1}^{n} V_{\alpha_i} = \bigcup_{i=1}^{n} V_{\alpha_i}^\circ \subseteq \bigcup_{i=1}^{n} V_{\alpha_i} \) \( \cdots \) (**), since \( V_{\alpha_i} \subseteq X \) for each \( \alpha_i \in \Omega \) and for each \( i=1,2,\ldots,n \). Then, \( \bigcup_{i=1}^{n} V_{\alpha_i} \subseteq X \) \( \cdots \) (**). From(**) and(**), we obtain that \( X = \bigcup_{i=1}^{n} V_{\alpha_i} \). Therefore, X is an r-compact space.

**3.3 Theorem:** Every nearly-compact space is an r-compact space.
Proof: Let $X$ be a nearly-compact space and let $\{ V_\alpha | \alpha \in \Omega \}$ be a regular open cover of $X$. From (1.1)part(ii), we obtain that $\{ V_\alpha | \alpha \in \Omega \}$ is an open cover of $X$. Since $X$ is a nearly-compact space, so there exist $V_\alpha_1, \ldots, V_\alpha_n$ such that $X = \bigcup_{i=1}^{n} V_{\alpha_i}$.

Since $\bigcup_{i=1}^{n} V_{\alpha_i} \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$, so, $X \subseteq \bigcup_{i=1}^{n} V_{\alpha_i} \ldots(*)$, since $V_{\alpha_i} \subseteq X$ for each $\alpha_i \in \Omega$ and for each $i=1,2,\ldots,n$. Then, $\bigcup_{i=1}^{n} V_{\alpha_i} \subseteq X \ldots(**)$. From(*) and(**), we obtain that $X = \bigcup_{i=1}^{n} V_{\alpha_i}$. Thus, $X$ is an $r$-compact space.

3.4 Theorem: Every quasi H-closed space is an $r$-compact space.
Proof: Let $X$ be a quasi H-closed space and let $\{ V_\alpha | \alpha \in \Omega \}$ be a regular open cover of $X$. From (1.1)part(ii), we conclude that $\{ V_\alpha | \alpha \in \Omega \}$ is an open cover of $X$. Since $X$ is quasi H-closed. So, there exist $V_\alpha_1, \ldots, V_\alpha_n$ such that $X = \bigcup_{i=1}^{n} V_{\alpha_i}$. Thus, $X$ is an $r$-compact space.

3.5 Theorem: Every S-closed space is an $r$-compact space.
Proof: Let $X$ be an S-closed space and let $\{ V_\alpha | \alpha \in \Omega \}$ be a regular open cover of $X$. From (1.1)part(iv), we can get that $\{ V_\alpha | \alpha \in \Omega \}$ is a semi open cover of $X$. Since $X$ is S-closed. So, there exist $V_\alpha_1, \ldots, V_\alpha_n$ such that $X = \bigcup_{i=1}^{n} V_{\alpha_i}$. Hence, $X$ is an $r$-compact space.

Directly from the definition of an extremely disconnected space, we can prove the following theorem:

3.6 Theorem: In an extremely disconnected space $X$, the following are equivalent:

i) $X$ is $r$-compact.
ii) $X$ is nearly-compact.
iii) $X$ is quasi H-closed.

4. Main Results

In this section, we prove several properties of $r$-compact spaces. First, we prove the finite intersection property in the following theorem:

4.1 Theorem: If a space $X$ is an $r$-compact and extremely disconnected space, then for every family $\{ V_\alpha | \alpha \in \Omega \}$ of regular closed sets in $X$ satisfying $\bigcap_{\alpha \in \Omega} V_\alpha = \varnothing$ there is a finite subfamily $V_{\alpha_1}, \ldots, V_{\alpha_n}$ with $\bigcap_{i=1}^{n} V_{\alpha_i} = \varnothing$.

Proof: Let $\{ V_\alpha | \alpha \in \Omega \}$ be a family of regular closed sets in $X$ satisfying $\bigcap_{\alpha \in \Omega} V_\alpha = \varnothing$. Then, $\{X - V_\alpha | \alpha \in \Omega \}$ is a regular open cover of $X$. Since $X$ is $r$-compact. Then, there exist a finite subfamily $V_{\alpha_1}, \ldots, V_{\alpha_n}$ such that :
\[ X = \bigcup_{i=1}^{n} X - V_{\alpha_i} = \bigcup_{i=1}^{n} X \cap V_{\alpha_i}^{c} = \bigcup_{i=1}^{n} V_{\alpha_i}^{c} \]. Since \( V_{\alpha_i}^{c} \) is a regular open set in \( X \), for each \( i=1,2,\ldots,n \) and for each \( \alpha_i \in \Omega \). So, \( V_{\alpha_i}^{c} \) is an open set, for each \( i=1,2,\ldots,n \) and for each \( \alpha_i \in \Omega \). And, since \( X \) is an extremely disconnected space, \( V_{\alpha_i}^{c} \) is an open set, for each \( i=1,2,\ldots,n \) and for each \( \alpha_i \in \Omega \). Hence, \( X = \bigcup_{i=1}^{n} V_{\alpha_i}^{c} \). Therefore, \( \bigcap_{i=1}^{n} V_{\alpha_i} = \emptyset \).

Next, we study the hereditary property of \( r \)-compact.

**4.2 Theorem:** Every regular closed subset of an \( r \)-compact is \( r \)-compact.

**Proof:** Let \( X \) be an \( r \)-compact space, \( F \) be a regular closed subset of \( X \) and let \( \{ V_{\alpha} \mid \alpha \in \Omega \} \) be a regular open cover of \( X \). Since \( F \) is a regular closed subset of \( X \), \( F^{c} \) is a regular open subset of \( X \). Thus, \( \{ V_{\alpha} \cup F^{c} \mid \alpha \in \Omega \} \) is a regular open cover of \( X \). Since \( X \) is an \( r \)-compact space, there exist \( V_{\alpha_1}, \ldots, V_{\alpha_n} \) such that \( X = \bigcup_{i=1}^{n} V_{\alpha_i} \cup F^{c} = \left( \bigcup_{i=1}^{n} V_{\alpha_i} \right) \bigcup F^{c} \). Then, \( F = F \cap X = F \cap \left[ \left( \bigcup_{i=1}^{n} V_{\alpha_i} \right) \bigcup F^{c} \right] = \left[ F \cap \left( \bigcup_{i=1}^{n} V_{\alpha_i} \right) \right] \bigcup \left[ F \cap F^{c} \right] \). So, \( F = F \cap \left( \bigcup_{i=1}^{n} V_{\alpha_i} \right) \). Hence, \( F \subseteq \left( \bigcup_{i=1}^{n} V_{\alpha_i} \right) \). That is, \( F = \left( \bigcup_{i=1}^{n} V_{\alpha_i} \right) \). So, \( F \) is an \( r \)-compact space.

Now, to study the continuous property we need for the following lemma:

**4.3 Lemma:** If \( f : X \to Y \) is a continuous function of a space \( X \) into an extremely disconnected space \( Y \), and if \( V \) is a regular open set in \( Y \), then \( f^{-1}(V) \) is a regular open set in \( X \).

**Proof:** Since \( V \) is a regular open set in \( Y \). Then, from (1.1) part(ii), we obtain that \( V \) is an open set in \( Y \), since \( f \) is continuous. So, \( f^{-1}(V) \) is an open set in \( X \). That is, \( [f^{-1}(V)]^{o} = f^{-1}(V) \) \((*)\). Now, since \( Y \) is an extremely disconnected space and since \( V \) is a regular open set in \( Y \). Thus, \( V = \overline{V} = V \). That is, \( V = \overline{V} \) \((**)\). Hence, \( [f^{-1}(V)]^{o} = [f^{-1}(V)]^{o} \) because of \( f \) is continuous. So, \( [f^{-1}(V)]^{o} = [f^{-1}(V)]^{o} \). From \((**)\), we can get that \( [f^{-1}(V)]^{o} = [f^{-1}(V)]^{o} \). From \((*)\) we conclude that, \( [f^{-1}(V)]^{o} = [f^{-1}(V)]^{o} \). Thus, \( [f^{-1}(V)]^{o} = f^{-1}(V) \) \((1)\). In another hand, we have
4.4 Theorem: If $f : X \to Y$ is a continuous function of an r-compact space $X$ onto an extremely disconnected space $Y$, then $Y$ is r-compact.

Proof: Let $\{ V_\alpha | \alpha \in \Omega \}$ be a regular open cover of $Y$. Since $f$ is a continuous function and $Y$ is an extremely disconnected space, then from the above lemma we obtain that $\{ f^{-1}(V_\alpha) | \alpha \in \Omega \}$ is a regular open cover of $X$. Since $X$ is r-compact, there exists a finite subfamily $f^{-1}(V_{\alpha_1}),..., f^{-1}(V_{\alpha_n})$ such that $X = \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$. Thus, $Y = f(\bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})) \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$. Hence, $Y = \bigcup_{i=1}^{n} V_{\alpha_i}$.

Directly, from (3.6) we can prove the following corollaries:

4.5 Corollary (1): If $f : X \to Y$ is a continuous function of an r-compact space $X$ onto an extremely disconnected space $Y$, then $Y$ is a quasi H-closed (nearly compact) space.

4.6 Corollary (2): If $f : X \to Y$ is a continuous function of a compact (or, nearly-compact, quasi H-closed, S-closed) space $X$ onto an extremely disconnected space $Y$, then $Y$ is an r-compact (quasi H-closed, nearly compact) space.

4.7 Corollary (3): If $f : X \to Y$ is a continuous function of an extremely disconnected and quasi H-closed (or, nearly compact) space $X$ onto an extremely disconnected space $Y$, then $Y$ is an r-compact (quasi H-closed, nearly compact) space.

Next, we study the converse continuous image of an r-compact space. So, we need the following lemma:

4.8 Lemma: Let $f : X \to Y$ be an open continuous bijective function of an extremely disconnected space $X$ into a space $Y$. If $V$ is a regular open set in $X$, then $f(V)$ is a regular open set in $Y$.

Proof: Since $V$ is a regular open set in $Y$ and since $X$ is an extremely disconnected space. Then, $V = \overline{f(V)}$. That is, $V = f(V)$. Since $f$ is a continuous function. So, $f(V) = f(\overline{V}) \subseteq \overline{f(V)}$. Since $V$ is an open function. Thus, $f(V) = [f(V)]^o \subseteq [f(V)]^o$.

That is, $f(V) \subseteq \overline{f(V)}$...(1). Now, we have $[f(V)]^o \subseteq [f(V)]$. Since $f$ is an open bijective function. So, $f$ is a closed function. Hence, $[f(V)]^o \subseteq [f(V)] = f(V)$.

Thus, $[f(V)]^o \subseteq f(V)$...(2). From (1) and (2) we can get that $[f(V)]^o = f(V)$. That is $f(V)$ is a regular open set in $Y$. ☐
**4.9 Theorem:** If \( f : X \rightarrow Y \) is an open continuous bijective function of an extremely disconnected space \( X \) onto an \( r \)-compact space \( Y \), then \( X \) is \( r \)-compact.

**Proof:** Let \( \{ V_\alpha \mid \alpha \in \Omega \} \) be a regular open cover of \( X \). From (4.8), we conclude that \( \{ f(V_\alpha) \mid \alpha \in \Omega \} \) is a regular open cover of \( Y \). Since \( Y \) is \( r \)-compact. Then, there exist \( f(V_{\alpha_1}), \ldots, f(V_{\alpha_n}) \) such that \( Y = \bigcup_{i=1}^{n} f(V_{\alpha_i}) \). Then, \( X = f^{-1}\left( \bigcup_{i=1}^{n} f(V_{\alpha_i}) \right) \). Since \( f \) is a closed function. Thus, \( f(V_{\alpha_i}) = f(V_{\alpha_i}) \). Thus,

\[
X = \bigcup_{i=1}^{n} f^{-1}\left( f(V_{\alpha_i}) \right) = \bigcup_{i=1}^{n} V_{\alpha_i}.
\]

Therefore, \( X \) is \( r \)-compact space.

From the above theorem and theorems (3.2), (3.3), (3.4), and (3.5) we can get the following corollary:

**4.10 Corollary(1):** If \( f : X \rightarrow Y \) is a homomorphism of an extremely disconnected space \( X \) onto a compact (or, nearly-compact, quasi H-closed, S-closed) space \( Y \), then \( Y \) is an \( r \)-compact space.

Directly from theorems (4.4) and (4.9) we can prove the following corollary:

**4.11 Corollary(2):** An \( r \)-compact property is a topological property under an extremely disconnected spaces.

**References**


