PreCartan G-Space

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Abstract
In this paper a preCartan G-space is our aim. Now we list the following some results that we have gotten: (i) a Cartan G-space is preCartan. (ii) We introduced some results on a net with a preopen set. (iii) We introduced this space (preCartan G-space) and give enough examples and theorems about it, where we study its properties, subspace, product, and the equivariant homeomorphic image.

Keywords: Preopen, preclosed, preneighborhood, precluster, preconvergence, strongly preopen function, preCartan G-space.

Introduction
The first step of studying preopen set was done in 1984 [5]. The authors were defined a set A to be preopen if \( A \subseteq \overline{A} \) and that the intersection of an open set and a preopen set is preopen.

The set of all preopen sets of a topological space \( X \) is denoted by \( PO(X) \), the complement of a preopen set is called preclosed [5]. The intersection of all preclosed sets containing \( A \) is called the preclosure of \( A \), denoted by \( \overline{A}^p \), which is the smallest preclosed set containing \( A \) [5], [11]. A subset \( A \) of \( X \) is called a preCartan G-space if \( A \subseteq \overline{A}^p \), which is the smallest preclosed set containing \( A \) [5], [11].

Preliminaries:
In this section, we recall the following theorems that we need:

**Theorem 2.1** [3]:
(i) A topological space \( X \) is \( T_2 \) if and only if every convergent net in \( X \) has a unique limit.
(ii) A topological space \( X \) is compact if and only if each net in \( X \) has a cluster point.
(iii) A net has \( y \) as cluster point if and only if it has a subnet which converges to \( y \).

**Theorem 2.2** [3]:
Let \( f \) be a function from a topological space \( X \) in to a topological space \( Y \).
Then \( f \) is continuous at \( x \in X \) if and only if whenever \( x_\alpha \rightarrow x \) in \( X \), then \( f(x_\alpha) \rightarrow f(x) \).

**Theorem 2.3** [9]:
Let \( X \) be a topological space and \( Y \subset X \). Then \( Y \) is open if and only if no net in \( X-Y \) can converge in a point in \( Y \).

**Theorem 2.4** [12]:
For each \( x \in X \), the isotropy subgroup \( G_x \) at \( x \) is closed.

**Theorem 2.5** [6]:
Let \( (X_i)_{i \in I} \) be a family of topological spaces and \( \emptyset \neq A_i \subseteq X_i \) for each \( i \in I \). Then \( \prod_{i \in I} A_i \) is preopen in \( \prod_{i \in I} X_i \) if and only if \( A_i \) is preopen in \( X_i \) for each \( i \in I \) and \( A_i \) is a non dense for only finitely many \( i \in I \).

**Theorem 2.6** [6]:
If \( U \) is a preopen subspace of a topological space \( X \), and \( V \) is a preopen subset of \( (U, \tau U) \), then \( V \) is preopen in \( X \).

**Theorem 2.7** [10]:
A subset \( A \) of a topological space \( X \) is preclosed set if and only if \( A = \overline{A}^p \).

**Theorem 2.8** [13]:
Let \( X \) be a topological space and \( A \subset X \), \( x \in X \). Then \( x \in \overline{A}^p \) if and only if there is a net \( (x_\alpha)_{\alpha \in A} \) in \( A \) such that \( x_\alpha \rightarrow x \).

PreCartan G-space:
A new G-space is introduced in this section which we call a preCartan G-space,
which is weaker than a Cartan G-space. But first we state and prove the following theorem.

**Theorem 3.1:**

Let \((x_\alpha)_{\alpha \in \Lambda}\) be a net in a topological space \(X\) such that \(x_\alpha \subset X, x \in X\) and let \(A \in PO(X)\) such that \(x \in A\). Then there exists a subnet \((x_\alpha_\mu)\) in \(A\) of the net \((x_\alpha)\) such that \(x_\alpha_\mu \to x\).

**Proof:**

Let \(U\) be an open subset of \(X\). Then \(U \cap A\) is a preopen set such that \(x \in U \cap A\). Let \(M = \{(\alpha, U \cap A)|\alpha \in \Lambda, U\ is\ an\ open\ subset\ of\ X, x \in U, \text{and} x_\alpha \in U \cap A\}\). Suppose that \(M\) is ordered as follows:

\((\alpha_1, U_1 \cap A) \leq (\alpha_2, U_2 \cap A)\) if and only if \(\alpha_1 \leq \alpha_2\) and \(U_1 \subseteq U_2\).

Clear that \(\leq\) is reflexive and transitive relations.

At the present time, let \((\alpha_1, U_1 \cap A)\) and \((\alpha_2, U_2 \cap A)\) be in \(M\).

\((U_1 \cap U_2) \cap A \in PO(X)\) and \(x \in (U_1 \cap U_2) \cap A\)

So \((x_\alpha)\) is frequently in \((U_1 \cap U_2) \cap A\).

Since \(\Lambda\) is a directed set and \(\alpha_1, \alpha_2 \in \Lambda\), then there exists \(\alpha_3 \in \Lambda\) such that \(\alpha_1 \leq \alpha_3\) and \(\alpha_2 \leq \alpha_3\).

Therefore, there exists \(\alpha_3 \in \Lambda\) such that \(x_\alpha_3 \in (U_1 \cap U_2) \cap A\).

i.e. \((\alpha_3, (U_1 \cap U_2) \cap A) \in M\) such that \(\alpha_1 \leq \alpha_3\), \(\alpha_2 \leq \alpha_3\) and \(U_1 \cap U_2 \subseteq U_1, U_1 \cap U_2 \subseteq U_2\).

Hence \((\alpha_1, U_1 \cap A) \leq (\alpha_3, (U_1 \cap U_2) \cap A)\) and \((\alpha_2, U_2 \cap A) \leq (\alpha_3, (U_1 \cap U_2) \cap A)\).

So \(M\) is a directed set.

Define \(g: M \to \Lambda\) such that \(g(\alpha, U \cap A) = \alpha\). To prove that \(xog\) satisfying a subnet conditions.

Let \((\alpha_1, U_1 \cap A) \leq (\alpha_2, U_2 \cap A)\). Then \(\alpha_1 \leq \alpha_2\) i.e. \(g(\alpha_1, U_1 \cap A) \leq g(\alpha_2, U_2 \cap A)\).

Let \(\alpha \in \Lambda\).

On the other hand, since \(X \cap A = A\) is a preopen subset of \(X\) which contains \(x\), then there exists \(\alpha' \in \Lambda\) such that \(x_\alpha' \in X \cap A\) and \(\alpha \leq \alpha'\).

So \((\alpha', X \cap A) \in M\), such that:

\(\alpha \leq \alpha' = g(\alpha', X \cap A)\)

Hence \(g\) defines a subnet of the net \((x_\alpha)\).

Now, let \(U_0\) be any open subset of \(X\) which contains \(x\).

Then \(U_0 \cap A\) is a preopen subset of \(X\) which contains \(x\).

We could find \(\alpha_0 \in \Lambda\) such that \(x_{\alpha_0} \in U_0 \cap A\).

So \((\alpha_0, U_0 \cap A) \in M\)

Hence for each \((\alpha, U \cap A) \in M\) and \((\alpha_0, U_0 \cap A) \leq (\alpha, U \cap A)\), we have \(\alpha_0 \leq \alpha\) and \(U \subseteq U_0\).

So \(x_0 \in U \subseteq U_0\).

This subnet is eventually in every neighborhood which contains \(x\).

Hence it is converges to \(x \in A\).

**Definition 3.2:**

A G-space \(X\) is called a preCartan G-space if every point of \(X\) has a thin preneighborhood.

**Example 3.3:**

(i) \((R, +)\) with the usual topology is a locally compact topological group, and the set:

\(D = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}\)

with the relative usual topology is a completely regular \(T_2\) space. Let \(R\) acts on \(D\) as follows:

\(\pi: R \times D \to D\) such that \(\pi(t, (x, y)) = (xe^t, ye^t)\) for each \(t \in R, (x, y) \in D\). Clear that \(D\) is an \(R\)-space.

To show that \(D\) is a preCartan \(R\)-space.

Let \((x, y) \in D\) and \(U = (x-\varepsilon, x+\varepsilon)\) be a preneighborhood of \(x\) in \(L = \{(x, 0) \in \mathbb{R}^2 \mid x \geq 0\}\), where \(\varepsilon\) does not equal neither \(x\) nor \(-x\). Let:

\(W = \{(0, y) \in \mathbb{R}^2 \mid y \geq 0\}\).

By theorem 2.5 we get that \(U \times W\) is a preneighborhood of \((x, y)\) in \(D\).

Before we prove \((U \times W, U \times W) = ((U, U))\), we need to show that \(W\) is an \(R\)-space and then we can continue solving the example.

\((W, +)\) with the relative usual topology is a topological group which is locally compact but not compact and \(R\) with the usual topology is a completely regular \(T_2\) space. Then \(R\) acts on \(W\) as follows:

\(\pi_1: R \times W \to W\), such that \(\pi_1(t, y) = ye^t\) for each \(y \in W, t \in R\). Clear that \(W\) is an \(R\)-space. Now to prove \(((U \times W, U \times W)) = ((U, U))\).

\(g \in ((U, U)) \leftrightarrow gU \cap U \neq \emptyset \leftrightarrow (gU \cap U) \times W \neq \emptyset \leftrightarrow gU \times W \cap U \times W \neq \emptyset \leftrightarrow g \in ((U \times W, U \times W))\).

Hence \(((U \times W, U \times W)) = ((U, U))\).

Yet we have to show that \(((U, U))\) has a compact closure.
\[ e^{-t}(x-\varepsilon) = x+\varepsilon \implies t_1 = \ln((x-\varepsilon)/(x+\varepsilon)) \]
\[ e^{-t_2}(x+\varepsilon) = x-\varepsilon \implies t_2 = \ln(((x+\varepsilon)/(x-\varepsilon)) \]

If \( x > 0 \), then \( t_1 = -t_2 \) and the set:
\[(U, U) = \{ g \in G \mid g \in U \not\in \emptyset \} = (-t_2, t_2) \]
has a compact closure. Hence \( D \) is a preCartan \( R \)-space.

![Fig. (I).](image)

(ii) \((R \setminus \{0\}, \cdot, )\) with the usual topology is a locally compact non-compact topological group. Besides, \( R^2 \) with the usual topology is a completely regular Hausdorff space. Then \( R\{0\} \) acts on \( R^2 \) as follows:
\[ \pi: R\{0\} \times R^2 \rightarrow R^2 \]
is defined by:
\[ \pi(r, (x, y)) = (rx, ry) \]
for each \( r \in R\{0\} \) and \((x, y) \in R^2 \).
Clear that \( R^2 \) is \( R\{0\} \)-space. But \( R^2 \) is not preCartan \( R\{0\}\)-space, since \((0, 0) \in R^2 \) has no thin preneighborhood since for any preneighborhood \( U \) of \((0, 0) \) the set \((U, U) = R\{0\} \) is not relatively compact in \( R\{0\} \).

**Proposition 3.4:**
A Cartan \( G \)-space is preCartan.

**Proof:**
Clear.

**Proposition 3.5:**
If \( X \) is a preCartan \( G \)-space, then:

a) Each orbit of \( X \) is preclosed.
b) For each \( x \in X \) the isotropy subgroup \( G_x \) at \( x \) is compact.

**Proof (a):**
Let \( X \) be a preCartan \( G \)-space to prove that \( G_x \) is preclosed in \( X \) (i.e., \( G_x = \overline{G_x}^p \)), we have to show that \( \overline{G_x}^p \subseteq G_x \).

Let \( y \in \overline{G_x}^p \). Then by 2.8, there is a net \((g_\alpha x) \) in \( G_x \) such that \( g_\alpha x \rightarrow x \).
Since \( X \) is preCartan, then there exists \( U \) a thin preneighborhood of \( y \).
By 3.1, there is a subnet \((g_\alpha x) \) of the net \((g_\alpha x) \) in \( U \) such that \( g_\alpha \rightarrow_y \).

Fixing \( \alpha_0 \), then \((g_\alpha x \rightarrow (g_\alpha x) = g_\alpha \rightarrow x \).

To prove \( g_\alpha \rightarrow (g_\alpha x) \in ((U, U)) \).
Because \((g_\alpha x) \) is in \( U \), then so is \((g_\alpha x) \).
Hence \((g_\alpha x \rightarrow (g_\alpha x) \) lies in \((g_\alpha \rightarrow (g_\alpha x) \) \)
U,
(i.e., \( U \cap (g_\alpha 0 \rightarrow (g_\alpha x)) \not\in \emptyset \).
Then \( g_\alpha \rightarrow (g_\alpha x) \not\in ((U, U)) \).
Since \((U, U)) \) is relatively compact, then by 2.1(ii), \((g_\alpha \rightarrow (g_\alpha x) \) has a cluster point say \( z \in G \).
Hence by 2.1(iii), we get that \((g_\alpha \rightarrow (g_\alpha x) \) has a subnet which converges to \( z \).
So \( g_\alpha \rightarrow g \), then \( g_\alpha \rightarrow g \).
By 2.2, we get that \( g_\alpha x \rightarrow g g \).
Since \( X \) is \( T_2 \), then by 2.1(i) we have \( y = g g \).
Hence \( G_x \subseteq \overline{G_x}^p \). But we have \( G_x \subseteq G_x \).
Therefore \( G_x \subseteq \overline{G_x}^p \). So by 2.7 we get that \( G_x \) is preclosed in \( X \).

b) Let \( x \in X \).
Since \( X \) preCartan, then there exists \( U \) a thin preneighborhood of \( x \).
The next step is to show that \( G_x \subseteq ((U, U)) \).
Let \( g \in G_x \) then \( g x = x \) which leads to \( g \not\in U \not\in \emptyset \).
Then \( g \in ((U, U)) \).
Hence \( G_x \subseteq ((U, U)) \) which is relatively compact and by 2.4 we get that \( G_x \) is closed in \( G \). Then \( G_x \) is compact.

**Proposition 3.6:**
If \( X \) is a preCartan \( G \)-space and \( x \in X \), then \( g \rightarrow g x \) is a preopen map of \( G \) onto \( G_x \).

**Proof:**
Let \( U \) be a preopen subset of \( G \).
To prove that \( U \cap G_x \) is preopen in \( G_x \) (i.e. \( (G-U) \cap G_x \) is preclosed in \( G_x \)).
Let \( y \in (G-U) \cap G_x \).
Then by 2.8, there is a net \((g_\alpha x) \) in \( (G-U) \cap G_x \) such that \( g_\alpha x \rightarrow y \).
Since \( X \) is preCartan, then there exists \( V \) a thin preneighborhood of \( y \).
By 3.1, there is a subnet \((g_\alpha \mu x)\) of the net \((g_\alpha x)\) in V such that \(g_\alpha \mu x \rightarrow y\).

Fixing \(\alpha_0\), then \((g_\alpha \mu g_\alpha^{-1}_0 (g_\alpha x) = g_\alpha \mu x)\).

As in the proof of 2.4(a), then \(g_\alpha \mu g_\alpha^{-1}_0 \in ((V, V))\).

Since \(((V, V))\) is relatively compact, then by 2.1(ii), \((g_\alpha \mu g_\alpha^{-1}_0)\) has a cluster point say \(g_\alpha\).

Hence by 2.1(iii), \((g_\alpha \mu g_\alpha^{-1}_0)\) has a subnet which converges to \(g_\alpha\).

So \(g_\alpha \mu g_\alpha^{-1}_0 \rightarrow g\), then \(g_\alpha \mu \rightarrow gg_\alpha\) and by 2.2, we get \(g_\alpha \mu x \rightarrow g \in g_\alpha x\).

Since \(U\) is open and \(g_\alpha \mu \not\in U\), then by 2.3, we have \(g \in g_\alpha \mu \not\in U\).

So \(g g_\alpha \mu \in G-U\).

Since \(X\) is \(T_2\), then by 2.1(i), we have \(y = gg_\alpha \mu x \in (G-U)x\).

Hence \((G-U)x \subseteq (G-U)x\).

But we have \((G-U)x \subseteq (G-U)x\).

Therefore \((G-U)x = (G-U)x\).

Then by 2.7, we get that \((G-U)x\) is preclosed. Hence \(Ux\) is preopen in \(Gx\).

**Theorem 3.7:**

Let \(X\) and \(Y\) be \(G\)-spaces and let \(\lambda: X \rightarrow Y\) be an onto, strongly preopen and equivariant function. If \(X\) is a semi Cartan \(G\)-space, then so is \(Y\).

**Proof:**

Let \(y \in Y\). Since \(\lambda\) is onto, then there exists \(x \in X\) such that \(\lambda(x) = y\).

Since \(X\) is a \(preCartan\) \(G\)-space and \(x \in X\), then \(x\) has \(U\) as a thin preneighborhood.

Since \(\lambda\) is strongly preopen, then \(\lambda(U)\) is a preneighborhood of \(y\). To show that \(\lambda(U)\) is thin we have to prove that \(((U, U)) = ((\lambda(U), \lambda(U)))\).

\(g \in ((U, U)) \Leftrightarrow g \in U \cap U \neq \emptyset \Leftrightarrow \lambda(g \in U \cap U) \neq \emptyset \Leftrightarrow \lambda \) is onto \(\lambda\) (\(g \notin \lambda(U) \neq \emptyset \Leftrightarrow \lambda \) is equivariant \(g \lambda(U) \cap \lambda(U) \neq \emptyset \Leftrightarrow g \in ((\lambda(U), \lambda(U)))\)). Hence:

\(((U, U)) = ((\lambda(U), \lambda(U)))\).

Because \(((U, U))\) is relatively compact, then so is \(((\lambda(U), \lambda(U)))\). Hence \(Y\) is a \(preCartan\) \(G\)-space.

**Proposition 3.8:**

If \(X\) is a \(preCartan\) \(G\)-space, \(H\) is a closed subgroup of \(G\) and \(Y\) is an \(open\) \(subspace\) of \(X\) which is an \(H\)-invariant \(subspace\) of \(X\), then \(Y\) is a \(preCartan\) \(H\)-space.

**Proof:**

By [1] \((H,Y)\) is a topological transformation group. Since \(Y\) is a \(subspace\) of \(X\) and \(X\) is a \(regular\) \(space\). Then so is \(Y\). Since \(G\) is \(locally\) \(compact\) and \(H\) is a \(closed\) subgroup of \(G\), then by [9] \(H\) is \(locally\) \(compact\). Hence \(Y\) is an \(H\)-space.

At the present time we are going to prove that \(Y\) is \(preCartan\). Let \(y \in Y\). Then \(y \in X\). Since \(X\) is a \(preCartan\) \(G\)-space then \(y\) has \(U\) as a thin \(preneighborhood\) in \(X\). Let \(U' = U \cap Y\).

Since \(Y\) is a \(preCartan\) \(H\)-space, then by 2.6 we get that \(U'\) is a \(preneighborhood\) of \(y\) in \(Y\).

So by [2] \(U'\) is a thin \(preneighborhood\) of \(y\) in \(Y\).

Hence \(Y\) is a \(preCartan\) \(H\)-space.

**Proposition 3.9:**

Let \(X\) and \(Y\) be \(G\)-spaces. Then \(X \times Y\) is a \(preCartan\) \(G\)-space if at least one of \(X\) or \(Y\) is \(preCartan\).

**Proof:**

At first we shall show that \(X \times Y\) is a \(G\)-space.

Since \(X\) is a \(G\)-space, then \(G\) acts on \(X\) by \(\pi_1: G \times X \rightarrow X\) such that \(\pi_1(g, x) = gx\) for each \(g \in G\) and \(x \in X\). Since \(Y\) is a \(G\)-space, then \(G\) acts on \(Y\) by \(\pi_2: G \times Y \rightarrow Y\) such that \(\pi_2(g, y) = gy\) for each \(g \in G\) and \(y \in Y\).

Define \(\pi: G \times X \times Y \rightarrow X \times Y\) such that:

\(\pi(g, (x, y)) = (gx, gy)\) for each \(g \in G\), \(x \in X\) and \(y \in Y\).

a) \(\pi\) is continuous.

b) \(\pi(e, (x, y)) = e(x, y) = (ex, ey) = (x, y)\)

c) \(\pi(g_1, \pi(g_2, (x, y))) = \pi(g_1, g_2(x, y))\)

Hence \(X \times Y\) is a \(G\)-space.

Now to prove that \(X \times Y\) is \(preCartan\).

Let \((x, y) \in X \times Y\).

Since \(x \in X\) and \(X\) is \(preCartan\), then there exists \(U\) a thin \(preneighborhood\) of \(x\).

By 2.5 we get \(U \times Y\) as a \(preneighborhood\) of \((x, y)\) in \(X \times Y\).
Because we have \(((U, U)) = ((U \times Y, U \times Y))\). So, \(((U \times Y, U \times Y))\) is relatively compact, which means that \(X \times Y\) is a preCartan G-space.

**Theorem 3.10:**

If a G-space \(X\) has a star thin preopen set \(U\), then \(X\) is a preCartan G-space.

**Proof:**

Let \(x \in X\).

Since \(U\) is a star set, then there is \(g \in G\) such that \(gx \in U\).

Hence \(x \in g^{-1} U\).

Since \(\pi_g: X \to X\) is strongly preopen for each \(g \in G\), then \(g^{-1} U\) is a preopen set of \(x\).

Since \(U\) is thin, then by [2] we get that \(((g^{-1} U, g^{-1} U))\) is relatively compact in \(G\).

That is \(g^{-1} U\) is a thin preneighborhood of \(x\) in \(X\)

Thus \(X\) is a preCartan G-space.

**Theorem 3.11:**

If \(X\) is a preCartan G-space, then:

(a) There is no fixed point.

(b) There is no periodic point.

**Proof:**

(a) Let \(x \in X\) such that \(x\) is a fixed point.

Since \(X\) is a preCartan G-space, then \(x\) has \(U\) as a thin preneighborhood in \(X\).

Because \(x\) is a fixed point, then \(gx = x\) for each \(g \in G\).

So \(gU \cap U \neq \emptyset\) for each \(g \in G\).

That is \(((U, U)) = G\).

Since \(((U, U))\) is relatively compact in \(G\), then \(G\) is compact.

But \(G\) is not compact, which leads to a contradiction.

Hence \(X\) has no fixed point.

(b) Let \(x \in X\) such that \(x\) is a periodic point.

Then \(G_x\) is a syndetic subgroup in \(G\).

That is there is a compact subset \(K\) of \(G\) such that \(G = G_x K\).

By 3.5(b) \(G_x\) is compact in \(G\) for each \(x \in X\).

Thus \(G\) is compact.

But that leads to a contradiction since \(G\) is not compact.

Hence \(X\) has no periodic point.

**References**


