The Coincidence Lefschetz Number For Self – Maps of Lie groups

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Abstract

Let $f, h : G \to G$ be any two self maps of a compact connected oriented Lie group G. In this paper, for each positive integer k, we associate an integer with f^k, h^k. We relate this number with Lefschetz coincidence number. We deduce that for any two differentiable maps $f, h : G \to G$, there exists a positive integer k such that $k \leq \lambda+1$, and there is a point $x \in G$ such that $f^k(x) = h^k(x)$, where λ is the rank of G.

Introduction

Let G be an n-dimensional compact connected Lie group with multiplication μ (i.e., $\mu : G \times G \to G$ such that $\mu(x, y) = x.y$) and unit e. Let $[G, G]$ be the set of homotopy classes of maps $G \to G$. Given two maps $f, f' : G \to G$, following [3], we write $f.f'$ to denote the map $G \to G$ defined by $(f.f')(g) = \mu(f(g), f'(g)) = f(g).f'(g)$, $g \in G$.

Given a point $g \in G$ and a differentiable map $F: G \to G$, write G_g to denote the tangent space of G at g [4,p.10], and denote by $d_g F$ the linear map $d_g F : T_g G \to T_{F(g)} G$ induced by F, it is called the differential of F at g [4,p.22]. Let $L_g, R_g : G \to G$ be respectively the left translation $L_g(g') = \mu(g, g')$, and the right translation $R_g(g') = \mu(g', g)$. Then there is a natural homomorphism Ad, the adjoin representation, from G to $GL(G_e)$, (the group of nonsingular linear transformations of G_e) defined as follows:-

$$Ad(g) = d_g R_{g^{-1}} \circ d_e L_g .$$

Note that $d_g R_{g^{-1}} \circ d_e L_g = d(R_{g^{-1}}(L_g(e))) \circ d_e L_g = d_e(R_{g^{-1}} \circ L_g) = d_e(L_g \circ R_{g^{-1}}) = d(L_g(R_{g^{-1}}(e))) \circ d_e R_{g^{-1}} = d_{g^{-1}} L_g \circ d_e R_{g^{-1}}$. Since G is connected, the image of Ad belongs to the connected component of $GL(G_e)$ containing the identity, i.e. for each $g \in G$, $\det Ad(g) > 0$. By Exercise A1

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[4,p. 147] we have

***Lemma* 1 :-**

1) If T is the map $G \to G$ is defined by $T(g) = g^{-1}$, then

$d_g T = -d_e L_{g^{-1}} \circ d_g R_{g^{-1}} = -d_e R_{g^{-1}} \circ d_g L_{g^{-1}}$,

2) If μ is the mapping $(g_1, g_2) \to g_1 g_2$ of $G \times G$ into G, then if $X \in G_{g_1}, Y \in G_{g_2}$.

$d_{(g_1,g_2)}\mu(X,Y) = d_{g_2} L_{g_1}(Y) + d_{g_1} R_{g_2}(X), (X,Y) \in G_{g_1} \times G_{g_2}$.

In [3], the author shows that if $f: G \to G$ is a differentiable map, then there exists a positive integer $k \leq \lambda + 1$ and a point $x \in G$ such that $f^k(x) = x$, where λ is the rank of G, i.e. the dimension of G of any maximal torus in G.

THE MAIN RESULTS

Let $f, h : G \to G$ be differentiable maps of compact connected oriented Lie group G. A point $g \in G$ is called a coincidence point if $f(g) = h(g)$, following [7],[8]. Assume f and h have isolated coincidence points then by compactness of G, f and h have only finitely many coincidence points. Also if g is an isolated coincidence point of f and h then $d_g f - d_g h$ has no nonzero fixed point, i.e. $\det(d_g f - d_g h) \neq 0$. In [6] the author defines the Lefschetz coincidence number as

$$Le(f,h) = \sum_{f(g)=h(g)} sign(\det(d_g f - d_g h)).$$

If f is the identity map then $Le(h)$ is the Lefschetz number, for more details see [2].

Let $Le : [G,G] \to Z$ be the function that sends each element in $[G,G]$ to its Lefschetz coincidence number. Then the Lefschetz coincidence point theorem states that "if f, $h : G \to G$ are maps with $Le(f,h) \neq 0$, then f, h have a coincidence point". Now we define another function $B(f,h) : [G,G] \to Z$ by setting $B(f,h) =$ degree$(f.h^{-1})$. Since $B(f,h) \neq 0$ implies $f.h^{-1}$ is surjective, i.e., $e \in \operatorname{Im}(f \circ h^{-1})$. this function also possesses the property that " if $B(f,h) \neq 0$, f and h have a coincidence point, i.e., if $B(f,h) \neq 0$ then $e \in \operatorname{Im}(fh^{-1})$ which means there is a point $g \in G$ such that $f(g)h^{-1}(g) = e$, i.e., $f(g) = h(g)$".

***Theorem* (2) :-**

The two functions Le, $(-1)^n B$: $[G,G] \to Z$ coincide, where n is the dimension of G.

proof :-

The left translation L_g of G onto itself is an analytic diffeomorphism then $d_e L_g$ is an isomorphism. So for each $g \in G$, we identify G_g with G_e by the differential of left translation for any L_g at e.

For any two maps in $[G,G]$, we take the representations $f, h: G \to G$ that satisfy the following :

(1) f and h are differentiable;

(2) f and h have only finitely many coincidence points $g_1, \ldots, g_k$, i.e., have isolated coincidence points $\{g_1, \ldots, g_k\}$.

(3) $\det(d_{g_i} h - d_{g_i} f) \neq 0$.

Then $Le(f,h) = \sum_{1}^{k} sign \det(d_{g_i} h - d_{g_i} f)$.