PAIRWISE SEPARATION AXIOMS IN
INTUITIONISTIC TOPOLOGICAL SPACES

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Abstract
The concept of intuitionistic topological space was introduced by Çoker. The aim of this paper is to generalize bitopological notions of separation to intuitionistic topological spaces.

الملخص
في هذا البحث قدمنا تعريف علاقات الفصل الزوجية للضاءات التبولوجية الثنائية وناقشنا العلاقة بين فضاء التبولوجي المضاعف وفضاء التبولوجي الثنائي من خلال تعميم هذه التعريف.

1-Introduction
After the introduction of fuzzy sets by Adamek [1], various mathematicians introduced generalization of the notion of fuzzy set. Among others, Atanassov [2, 3] introduced the notion of intuitionistic fuzzy set. Chang [7] used fuzzy sets to introduce the concept of a fuzzy topology. Later this concept was extended to intuitionistic fuzzy topological spaces by Çoker in [8]. The concept of intuitionistic set, which is a classical version of an intuitionistic fuzzy set, was first given by Çoker in [9]. He studied topology on intuitionistic sets in [10]. In this context, Coker et al. [9, 13] studied continuity, connectedness, compactness and separation axioms in intuitionistic fuzzy topological spaces. In this paper we follow the suggestion of J.G. Garcia and S.E. Rodabaugh that (double-fuzzy set) is a more appropriate name than (intuitionistic fuzzy set), and therefore adopt the term (double-set) for the intuitionistic set, and (double-topology) for the intuitionistic topology of Dogan Çoker, (this issue) we denote by Dbl-Top the construct (concrete texture over Set) whose objects are pairs \((X, \tau)\) where \(\tau\) is a double-topology on \(X\). In Section 4 we discuss the relations between these bitopologically based on concepts and some previously defined separation axioms for double-topological spaces.

2-Preliminaries
Throughout the paper by \(X\) we denote a non-empty set. In this section we shall present various fundamental definitions and propositions. The following definition is obviously inspired by Atanassov [3].

2.1. Definition. [8] A double-set (DS in brief) \(A\) is an object having the form \(A=\langle x, A_1, A_2 \rangle\), Where \(A_1\) and \(A_2\) are subsets of \(X\) satisfying \(A_1 \cap A_2 = \emptyset\). The set \(A_1\) is called the set of members of \(A\), while \(A_2\) is called the set of non-members of \(A\).

Throughout the remainder of this paper we use the simpler \(A=(A_1, A_2)\) for a double-set.

2.2. Remark. Every subset \(A\) of \(X\) may obviously be regarded as a double-set having the form \(A=(A, A^c)\), where \(A^c= X-A\) is the complement of \(A\) in \(X\).

we recall several relations and operations between DS’s as follows:

2.3. Definition. [8] Let the DS’s \(A\) and \(B\) on \(X\) be the form \(A=(A_1, A_2)\), \(B=(B_1, B_2)\), respectively. Furthermore, let \(\{A_j : j \in J\}\) be an arbitrary family of DS’s in \(X\), where \(A_j = (A_j^1, A_j^2)\). Then
(a) \( A \subseteq B \) if and only if \( A_1 \subseteq B_1 \) and \( A_2 \supseteq B_2 \);
(b) \( A = B \) if and only if \( A \subseteq B \) and \( B \subseteq A \);
(c) \( \bar{A} = (A_2, A_1) \) denotes the complement of \( A \);
(d) \( \bigcap A_j = (\bigcap A_j^{(1)}, \bigcup A_j^{(2)}) \);
(e) \( \bigcup A_j = (\bigcup A_j^{(1)}, \bigcap A_j^{(2)}) \);
(f) \( [\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!] A = (A_1, A_2) \);
(g) \( \langle A \rangle = (A_2^c, A_2) \);
(h) \( \phi = (\phi, X) \) and \( X = (X, \phi) \).

In this paper we require the following:
(i) \( \langle A \rangle = (A_1, \phi) \) and \( \langle A \rangle = (\phi, A_2) \).

Now we recall the image and preimage of DS’ under a function.

**2.4. Definition.** [8,13] Let \( x \in X \) be a fixed element in \( X \). Then:
(a) The DS given by \( x = (\{ x \}, \{ x \}^c) \) is called a double–point (DP in brief \( X \)).
(b) The DS \( \bar{x} = (\phi, \{ x \}^c) \) is called a vanishing double-point (VDP in brief \( X \)).

**2.5. Definition.** [8,13]
(a) Let \( x \) be a DP in \( X \) and \( A= (A_1, A_2) \) be a DS in \( X \). Then \( x \in A \) iff \( x \in A_1 \).
(b) Let \( x \) be a VDP in \( X \) and \( A= (A_1, A_2) \) a DS in \( X \). Then \( x \in A \) iff \( x \notin A_2 \).

It is clear that \( x \in A \Leftrightarrow x \subseteq A \) and that \( \bar{x} \in A \Leftrightarrow x \subseteq A \).

**2.6. Definition.** [10] A double-topology (DT in brief) on a set \( X \) is a family \( \tau \) of DS’ in \( X \) satisfying the following axioms:

T1: \( \phi, X \in \tau \),
T2: \( G_1 \cap G_2 \in \tau \) for any \( G_1, G_2 \in \tau \),
T3: \( \bigcup G_j \in \tau \) for any arbitrary family \( \{ G_j : j \in J \} \subseteq \tau \).

In this case the pair \( (X, \tau) \) is called a double-topological space (DTS in brief), and any DS in \( \tau \) is known as a double open set (DOS in brief). The complement \( \bar{A} \) of a DOS \( A \) in a DTS is called a double closed set (DCS in brief) in \( X \).

**2.7. Definition.** [10] Let \( (X, \tau) \) be an DTS and \( A = (A_1, A_2) \) be a DS in \( X \).

Then the interior and closure of \( A \) are defined by:
\[
\text{int}(A) = \bigcup \{ G : G \text{ is a DOS in } X \text{ and } G \subseteq A \},
\]
\[
\text{cl}(A) = \bigcap \{ H : H \text{ is a DCS in } X \text{ and } A \subseteq H \},
\]
respectively.

It is clear that \( \text{cl}(A) \) is a DCS in \( X \) and \( \text{int}(A) \) a DOS in \( X \). Moreover, \( A \) is a DCS in \( X \) iff \( \text{cl}(A) = A \), and \( A \) is a DOS in \( X \) iff \( \text{int}(A) = A \).

**2.8. Example.** [10] Any topological space \( (X, \tau_0) \) gives rise to a DT of the form
\( \tau = \{ A : A \in \tau_0 \} \) by identifying a subset \( A \) in \( X \) with its counterpart \( \bar{A} = (A_1, A_2^c) \), as in Remark 2.2.

3- The Constructs Dbl-Top and Bitop:
We begin recalling the following result which associates a bitopology with a double topology.

**3.1. Proposition.** [10] Let \( (X, \tau) \) be a DTS.
(a) \( \tau_1 = \{ A_1 : \exists A_2 \subseteq X \text{ with } A = (A_1, A_2) \in \tau \} \) is a topology on \( X \).
(b) $\tau^*_2 = \{ A_2 : \exists A_1 \subseteq X \; \text{with} \; A = (A_1, A_2) \in \tau_1 \}$ is the family of closed sets of the topology $\tau_2 = \{ A_2 : \exists A_1 \subseteq X \; \text{with} \; A = (A_1, A_2) \in \tau \}$ on $X$.

(c) Using (a) and (b) we may conclude that $(X, \tau_1, \tau_2)$ is a bitopological space.

3.2. Proposition. Let $(X, u, v)$ be a bitopological space. Then the family

$$\{(U, V^c) : U \subseteq u, V \subseteq v, U \subseteq V\}$$

is a double topology on $X$.

Proof. The condition $U \subseteq V$ ensures that $U \cap V^c = \emptyset$, while the given family contains $\emptyset$ because $\emptyset \subseteq u, v$, and it contains $X$ because $X \subseteq u, v$. Finally this family is closed under finite intersections and arbitrary unions by Definition 2.3 (d,e) and the corresponding properties of the topologies $u$ and $v$.

3.3. Definition. Let $(X, u, v)$ be a bitopological space. Then we set

$$\tau_{uv} = \{(U, V^c) : U \subseteq u, V \subseteq v, U \subseteq V\}$$

and call this the double topology on $X$ associated with $(X, u, v)$.

3.4. Proposition. If $(X, u, v)$ is a bitopological space and $\tau_{uv}$ the corresponding DT on $X$, then

$$\tau_{uv} = \{ (u, \emptyset) \} = u \text{ and } (\tau_{uv})_2 = v.$$ 

Proof. $U \subseteq u$ implies $(U, \emptyset) \in \tau_{uv}$ since $U \subseteq X \in v, sou \subseteq (\tau_{uv})_1$. Conversely, take $U \in (\tau_{uv})_1$. Then $(U, B) \in \tau_{uv}$ for some $B \subseteq X$, and now $U \subseteq u$. Hence $(\tau_{uv})_1 \subseteq u$, and the first equality is proved.

The proof of the second equality may be obtained in a similar way, and we omit the details.

4. Pairwise Separation Axioms in Double Topological Space.

In this section we use the link between bitopological spaces and double topological spaces established above to generalize several bitopological separation properties to the case of double topological spaces. We limit ourselves to the strong forms of the pairwise $T_3$, $T_4$ properties and we will also consider the pairwise $R_3$ and $R_4$ axioms.

The (strong) $T_0$ and $T_1$ axioms are due to Fletcher, Hoyle and Patty [11], pairwise $T_2$ due to Kelly [12]:

4.1. Definition. [12] In a bitopological space $(X, u, v)$, $u$ is said to be $T_3$ with respect to $v$ if

$$\forall DCS A \subseteq u, \forall a \in \text{int} A \in X, \exists B \subseteq u, D \subseteq v \text{ and } a \in B,$$

$$A \subseteq D \text{ and } B \cap D = \emptyset.$$ 

$(X, u, v)$ is said to be pairwise $T_3$, if $u$ is $T_3$ with respect to $v$ and $v$ is $T_3$ with respect to $u$.

4.2. Proposition. If $(X, u, v)$ is pairwise $T_3$ then $(X, \tau_{uv})$ satisfies the following condition:

Given $DCS A \subseteq X$, $A \subseteq u$, $a \in \text{int} A$ there exists $G, H \in \tau_{uv}$ with $a \in G, a \notin H$,

$$A \subseteq H \text{ and } G \subseteq (\overline{H}).$$

Proof. Let $(X, u, v)$ be pairwise $T_3$ and $DCS A \subseteq X$, such that $A \subseteq u, a \in \text{int} A$, then we have $B \subseteq u$ and $D \subseteq v$ and $a \in B, A \subseteq D$ and $B \cap D = \emptyset$ that is $B \subseteq D^c$.

Now defining $G = (B, \emptyset), H = (\emptyset, D^c)$ gives $G, H \in \tau_{uv}$, $a \in G$ and $a \in D^c$

then $a \notin H, A \subseteq D^c$ so $A \subseteq H$ and $G = (B, \emptyset) \subseteq (\overline{H}) = (D^c, \emptyset)$.

4.3. Definition. The DTS $(X, \tau)$ is called pairwise $T_3$ if $\forall DCS A \in \tau, a \in \text{int} A \in X$ there exists $G, H \in \tau$ satisfying $a \in G, a \notin H$.

$A \subseteq H$ and $G \subseteq (\overline{H})$.
4.4. Proposition. If \((X, \tau)\) is pairwise \(T_3\) then \((X, \tau_1, \tau_2)\) is pairwise \(T_1\).

Proof. Let \(A\) be a DCS in \(X, a \in \text{int } A\ , \exists G = (B, C), H = (D, F) \in \tau\) with
\[ a \in G, a \in H, A \subseteq H \text{ and } G \subseteq (\bar{H}) \text{ clearly } a \in B \in \tau_1 \ , \text{ from } G \subseteq (\bar{H}) = (F, \phi) \]
\[ , B \subseteq F \text{ and since } a \not\in H \text{ then } a \in F \Rightarrow a \not\in F^c \in \tau_2 \text{ and } A \subseteq H \text{ where } A = (A_1, A_2) \]
then \(A_1 \subseteq D, F \subseteq A_2\) , let \(F^c = D\) then \(A_1 \subseteq F^c \) and \(B \cap F^c = \phi\) , we prove that \(\tau_1\) is \(T_3\) with respect to \(\tau_2\) , in the same way we prove \(\tau_2\) is \(T_3\) with respect to \(\tau_1\) , then \((X, \tau_1, \tau_2)\) is pairwise \(T_3\) .  

\[ \square \]

4.5. Corollary. \((X, u, v)\) is pairwise \(T_3\) iff \((X, \tau_{uv})\) is pairwise \(T_3\).

Proof. Necessity follows from Proposition 4.2 and sufficiency from proposition 4.4 and proposition 3.4 .  

\[ \square \]

4.6. Definition. In a bitopological space \((X, u, v)\) , \(u\) is said to be \(R_3\) with respect to \(v\) , if
for every DCS \(A \in u, a \in \text{int } A \) in \(X\) and \(a \not\in \text{ cl}_u(A), A \not\subseteq \text{ cl}_v(\{a\})\) then
\[ \exists B \in u, D \in v \text{ such that } a \in B, A \subseteq D, B \cap D = \phi \ . \ (X, u, v)\) is pairwise \(R_3\) , if \(u\) is \(R_3\) with respect to \(v\) and \(v\) is \(R_3\) with respect to \(u\) .

4.7. Proposition. If \((X, u, v)\) \(R_3\) then \((X, \tau_{uv})\) satisfies the condition : \(\forall \text{ DCS } A, a \in \text{ int } A \text{ in } X\) , if there exists \(M \in \tau_{uv}\) with \(a \in M, A \not\subseteq M\ (A \subseteq M, a \in M)\) then there exists
\[ G, H \in \tau_{uv}\text{ such that } a \in G, A \subseteq H \text{ and } G \subseteq (\bar{H}) . \]

Proof. If \(a \in M, A \not\subseteq M\) then \(M = (P, Q), a \in P \in u, A = (A_1, A_2) \not\subseteq M\),
\[ A_1 \not\subseteq P \text{ so } a \not\in \text{ cl}_u(A) \text{ likewise } A \subseteq M, a \in M \Rightarrow A_1 \subseteq P \text{ and } a \not\in Q \in v \text{ so} \]
\[ A \not\subseteq \text{ cl}_v(\{a\}) \text{ so in either case we have } B \in u, D \in v \text{ satisfying} \]
\[ a \in B, A \subseteq D \text{ and } B \cap D = \phi \text{ that is } B \subseteq D^c . \text{ Now defining} \]
\[ G = (B, \phi), H = (\phi, D^c) \text{ gives } G, H \in \tau_{uv}, a \in G, A \not\subseteq D^c \Rightarrow A \subseteq H, \text{ and } G \subseteq (\bar{H}) . \]

\[ \square \]

4.8. Definition. The DTS \((X, \tau)\) is called pairwise \(R_3\) , if it satisfies the condition : \(\forall \text{ DCS } A, a \in \text{ int } A \text{ in } X\) , if there exists \(M \in \tau\) with \(a \in M\),
\[ A \not\subseteq M\ (A \subseteq M, a \in M)\ then \text{ there exists } G, H \in \tau_{uv}\text{ such that } a \in G, A \subseteq H \text{ and } G \subseteq (\bar{H}) . \]

4.9. Proposition. If \((X, \tau)\) is pairwise \(R_3\) then \((X, \tau_1, \tau_2)\) pairwise \(R_3\).

Proof. First to prove \(\tau_1\) is \(R_3\) with respect to \(\tau_2\) take a DCS \(A, a \in \text{ int } A \text{ in } X\) with \(a \not\in \text{ cl}_{\tau_1}(A)\).
Now we have \(a \in P \in \tau_1\) with \(A_1 \not\subseteq P, A = (A_1, A_2) \text{ so for some } G \subseteq X\) we have
\(P, Q) \in \tau . \text{ sitting } M = (P, Q) \text{ gives } a \in M, A \not\subseteq M \text{ so we have } G, H \in \tau \text{ satisfying} \]
\[ a \in G, A \subseteq H \text{ and } G \subseteq (\bar{H}) . \text{ By a similar argument } G, H \text{ satisfying the same conditions also be} \]
obtain if \(A \not\subseteq \text{ cl}_{\tau_2}(\{a\})\) the existence of \(\tau_1, \tau_2\) open sets separating \(a\) and \(A\) as follow ,
let \(G = (B, C), H = (D, F) \text{ with } a \in G, A \subseteq H, A = (A_1, A_2) \text{ clearly } a \in B \in \tau_1\)
\[ A \subseteq H \text{ then } A_1 \subseteq D, F \subseteq A_2 \text{ so } A_1 \not\subseteq F\]
\[ \Rightarrow A_1 \subseteq F^c \in \tau_2 \text{ while } (B, C) \subseteq (\bar{H}) = (F, \phi) \]
\[ \Rightarrow B \subseteq F \text{ so that } C \cap F^c = \phi . \text{ in a same way we prove that } \tau_2 \text{ is } R_3 \text{ with respect to } \tau_1 \text{ then} \]
\((X, \tau_1, \tau_2)\) pair wise \(R_3\) .  

\[ \square \]
4.11. Corollary. (X,u,v) is pairwise R₃ iff (X,τᵥ) is pairwise R₃.

Proof. Necessity follows from proposition 4.7 and sufficiency from proposition 4.9 and proposition 3.4.

4.12. Definition. The bitopological space (X,u,v) is called pairwise T₄ if for every two DCS A and B in X

and A\cap B = \emptyset then there exist G \subseteq u, H \subseteq v and A \subseteq G, B \subseteq H, G \cap H = \emptyset.

4.13. Proposition. If (X,u,v) is pairwise T₄ then (X,τᵥ) has the property: Given A , B as DCS's in X , there exists N,M \in τᵥ with A \subseteq N, B \subseteq M and N \subseteq \overline{M}.

Proof. Take A,B DCS in X, s.t A\cap B = \emptyset and G \subseteq u, H \subseteq v satisfying

A \subseteq G, B \subseteq H and G \cap H = \emptyset which is G \subseteq H^c then A \subseteq H^c.

Now defining N = (G,φ), M = (φ,H^c) gives N,M \in τᵥ, A \subseteq N, B \subseteq H then

B \subseteq H so that B \not\subseteq H^c \Rightarrow B \subseteq M and N \subseteq \overline{M}.

4.14. Definition.[12] The DTS (X,τ) is called pairwise T₄ if

\forall DCS A,B in X there exists N,M \in τ satisfying A \subseteq N, B \subseteq M and N \subseteq \overline{M}.

4.15. Proposition. If (X,τ) is pairwise T₄ then (X,τ₁,τ₂) is pairwise T₄.

Proof. \forall DCS A,B in X , take N = (C,D), M = (G,H) \in τ with

A \subseteq N \in τ₁, B \subseteq M and N \subseteq \overline{M}. clearly

A_i \subseteq C \subseteq τ₁ for A = (A_i,A_2) and B \subseteq M then B \not\subseteq \overline{M} and B \subseteq H.

for B = (B_i,B_2) then B_i \subseteq H^c \in τ₂ and

N \subseteq \overline{M} then (C,D) \subseteq (H,φ) \Rightarrow C \subseteq H so C \cap H^c = \emptyset.

4.16. Corollary. (X,u,v) is pairwise T₄ iff (X,τᵥ) is pairwise T₄.

Proof. Necessity follows from proposition 4.13 and sufficiency from proposition 4.15 and proposition 3.4.

4.17. Definition. The bitopological space (X,u,v) is called pairwise R₄ if

\forall DCS A, Bin X, A\cap B = \emptyset, A \subset \text{cl}_u(B), B \subset \text{cl}_v(A) implies there exists G \subseteq u, H \subseteq v

with G \cap H = \emptyset and A \subseteq G, B \subseteq H.

4.18. Proposition. If (X,u,v) is pairwise R₄ then (X,τᵥ) satisfies the condition:

\forall DCS A, B in X

A\cap B = \emptyset if there exists M \in τᵥ with A \subseteq M, B \subseteq M (A \not\subseteq M, B \subseteq M)

Then there exists G,H \in τᵥ s.t A \subseteq G, B \subseteq H and G \subseteq \overline{H}.

Proof. If we have M \in τᵥ with

A \subseteq M & B \not\subseteq M then M = (P,Q), A = (A_1,A_2) \subseteq M, A_1 \subseteq P \subseteq u, Q \subseteq A_2, B = (B_1,B_2) \subseteq M, B_1 \not\subseteq P \not\subseteq B_2 so A \not\subseteq \text{cl}_u(B). Likewise

B \subseteq M, A \not\subseteq M, B \subseteq P & A_1 \not\subseteq P so B \not\subseteq \text{cl}_v(A), so in either case we have

N_1 \subseteq u, N_2 \subseteq v satisfying A \subseteq N_1, B \subseteq N_2 & N_1 \cap N_2 = \emptyset that is N_1 \subseteq N_2

Now defining G = (N_1,φ), H = (φ,N_2) gives G,H \in τᵥ, A \subseteq G, B \subseteq H

and G \subseteq \overline{H}.

4.19. Definition. The DTS (X,τ) is called pairwise R₄ if it satisfies the condition: \forall DCS A,B in X

A\cap B = \emptyset if there exists M \in τᵥ with A \subseteq M, B \not\subseteq M (A \not\subseteq M, B \subseteq M)
Then there exists \( G,H \in \tau_{uv} \) s.t \( A \subseteq G,B \subseteq H \) and \( G \subseteq (\overline{H}) \).

**4.20. Proposition.** If \((X,\tau)\) is pairwise \(R_4\) then \((X,\tau_1,\tau_2)\) is pairwise \(R_4\).

**Proof.** First take two DCS \( A \) and \( B \) in \( X \) with \( A \not\subseteq cl_{\tau_i}(B) \). Now we have

\[
A \subseteq P \in \tau_1 \text{ for } A = (A_1,A_2) \text{ so for some } Q \subseteq X \text{ we have } (P,Q) \in \tau
\]

Choosing \( M = (P,Q) \) gives \( A \subseteq M,B \not\subseteq M \) so we have \( G,H \in \tau \) satisfying

\[
A \subseteq G,B \subseteq H \text{ and } G \subseteq (\overline{H}). \text{ By a similar argument } G,H \text{ satisfying the same conditions may also be obtain if } B \not\subseteq cl_{\tau_i}(A) \text{ existence of } \tau_1,\tau_2 \text{ open }
\]

Sets. Separating \( A \) and \( B \) as follow

Let \( G = (D,C),H = (F,L) \) with \( A \subseteq G,B \subseteq H \) clearly \( A_i \subseteq D \in \tau_1, B \subseteq H \) so that

\[
B_i \subseteq F,L \subseteq B \text{ then } B_i \not\subseteq L \text{ so } B_i \subseteq L' \in \tau_2 \text{ while } (D,C) \subseteq (\overline{H}) = (L,\phi)
\]

Then \( D \subseteq L \) and \( D \cap L' = \phi \).

\( \square \)

**4.21. Corollary.** \((X,u,v)\) is pairwise \(R_4\) iff \((X,\tau_{uv})\) is pairwise \(R_4\).

**Proof.** Necessity follows from proposition 4.18 and sufficiency from proposition 4.20 and proposition 3.4.

**References**


