Lie and Jordan Structure in Prime $\Gamma$-rings with $\Gamma$-centralizing Derivations

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Abstract
Let $M$ be a prime $\Gamma$-ring satisfying $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$ with center $Z$, and $U$ be a Lie (Jordan) ideal. A mapping $d : M \rightarrow M$ is called $\Gamma$-centralizing if $[u, d(u)]_\alpha \in Z$ for all $u \in U$ and $\alpha \in \Gamma$. In this paper, we studied Lie and Jordan ideal in a prime $\Gamma$-ring $M$ together with $\Gamma$-centralizing derivations on $U$.

Keywords: Prime $\Gamma$-ring, Lie ideal, Jordan ideal, $\Gamma$-centralizing, Derivation.

1. Introduction
N. Nobusawa [1] introduced the notion of $\Gamma$-ring, more general than a ring. W. E. Barnes [2] weakened slightly the conditions in the definition of $\Gamma$-ring in the sense of Nobusawa after these two papers were published, number of modern algebraists have determined a lot of fundamental properties of $\Gamma$-ring and extended numerous significant results in classical ring theory to gamma ring theory see [3, 4, 5 and 6] for partial references.

In classical ring the theory of centralizing mapping on prime ring was initiated by Posner [7] who proved that the existence of a nonzero derivation on a prime ring forces the ring to be commutative. In [8] R. Awtar considered centralizing derivations on Lie and Jordan ideals generalized Posner's theorem. A lot of work has been done during the last decades in this field see [9, 10, 11, and 12] where further reference can be found.

By the same motivation as in the classical ring theories we proved the following results. Let $M$ be a prime $\Gamma$-ring, satisfying, $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$ and it will represented by (*)

i) If characteristic of $M$ is different from 2 and 3 and $U$ be Lie ideal then if $d$ is $\Gamma$-centralizing
on \( U \) then \( U \) is central in \( M \).

ii) If \( M \) has characteristic \( 3 \) and \( U \) is Jordan ideal, then if \( d \) is \( \Gamma \) -centralizing then \( U \) is central in \( M \) further, if \( U \) is a Lie ideal with \( uca \in U \) for all \( u \in U \) and, \( \alpha \in \Gamma \), then \( U \) is central in \( M \). The case when \( M \) has characteristic \( 2 \) is also studied.

2. Some Basic Definitions

**Definition 2.1** [2]: Let \( M \) and \( \Gamma \) be two additive abelian groups. If there exists a mapping \((a,a),b \mapsto aab \) of \( M \times \Gamma \times M \rightarrow M \) which satisfies for all \( a,b,c \in M \) and \( \alpha, \beta \in \Gamma \):

1) \((a+b)\alpha = aac + bca,
2) \alpha(a + \beta)b = aab + ab\beta,
3) \alpha b + c = abc + aac.

\( (aabb) \beta \) is called \( \Gamma \)-ring in the sense of Barnes.

**Definition 2.2** [3]: An additive subgroup \( S \) of a \( \Gamma \)-ring \( M \) is called subring if \( \Sigma S \in S \).

**Definition 2.3** [3]: An additive subgroup \( I \) of \( M \) is said to be a left (or right) ideal of \( M \) if \( M \Gamma I \subseteq I \) (or \( I \Gamma M \subseteq I \)), if \( I \) is both a right and left ideal, then we say that \( I \) is an ideal.

**Definition 2.4** [3]: Let \( M \) be a \( \Gamma \)-ring then \( M \) is called prime if \( a, b \in M \) and \( a \Gamma M b = 0 \) implies either \( a = 0 \) or \( b = 0 \) where \( a,b \in M \).

**Definition 2.5** [3]: A subset \( S \) of a \( \Gamma \)-ring \( M \) is called strongly nilpotent if there exists a positive integer \( n \) such that \( \Sigma \Gamma^n S = 0 \).

**Remark:**
1) For any \( a, b \in M \), \( ab \beta - bca \) are denoted by \([a,b]_\alpha \). Then one has the basic identities,

\([a, b\alpha]_\beta = [a, \alpha]_\beta b + a\beta b, [a, \beta]_\alpha b, [a, \beta]_\alpha \),

And,

\([a, b\beta]_\alpha = b\beta [a, \alpha] + [a, \alpha] + b\beta [a, \alpha], [a, b\beta]_\alpha \),

for all \( a, b, c \in M \) and \( \alpha, \beta \in \Gamma \). Using the assumption (*) the above identities reduce to,

\([a, b\beta]_\alpha = [a, \alpha]_\beta b + a\beta b, [a, \beta]_\alpha \).

And,

\([a, b\beta c]_\alpha = b\beta [a, \alpha]_\beta + [a, \alpha] + b\beta [a, \alpha], [a, b\beta c]_\alpha \).

2) Let \( M \) be a \( \Gamma \)-ring, the center of \( M \) is defined as, \( Z = \{a \in M : a\alpha \alpha = a\alpha \alpha \text{ for all } m \in M, \alpha \in \Gamma \} \).

**Definition 2.6** [13]: An additive subgroup \( U \) of a \( \Gamma \)-ring \( M \) is said to be a Lie ideal of \( M \) if \([u, m]_\alpha \in U, \text{ for all } u \in U, m \in M \) and \( \alpha \in \Gamma \). And \( U \) is said to be Jordan ideal if \( uca + mac \in U \), for all \( u \in U, m \in M \) and \( \alpha \in \Gamma \).

**Definition 2.8** [14]: An additive mapping \( d : M \rightarrow M \) is called a derivation of \( M \) if,

\( d(xm) = d(x\alpha) + xadm(x), \text{ holds for all } x, y \in M \) and \( \alpha \in \Gamma \).

For a fixed \( a \in M \) and \( \alpha \in \Gamma \) the mapping, \( I^\alpha : M \rightarrow M \) given by \( I^\alpha = [m, a]_\alpha \), is said to be inner derivation of \( M \) [15].

**Definition 2.9** [16]: Let \( M \) be a \( \Gamma \)-ring with center \( Z \) and \( U \) be lie (Jordan) ideal of \( M \). A mapping \( d : M \rightarrow M \) is called \( \Gamma \)-centralizing (resp. \( \Gamma \)-commuting) if \([u, d(u)]_\alpha \in Z \) (resp. \([a, d(u)]_\alpha = 0, \text{ for all } u \in U, \alpha \in \Gamma \).

3. Basic Lemmas

For proving our main results, we need some important results which we have proved here as lemmas. So, we start as follows:

**Lemma 3.1**: Let \( M \) be a prime \( \Gamma \)-ring, \( d \) a nonzero derivation of \( M \) and \( a \) be an element of \( M \) if \( aacd(m) = 0 \), for all \( m \in M \) and \( \alpha \in \Gamma \). Then either \( a = 0 \) or \( d \) is zero.

**Proof:**
We have \( aacd(m) = 0 \), for all \( m \in M \) and \( \alpha \in \Gamma \). Replace \( m \) by \( maks \) where \( x \in M \), then

\( aacd(maks) = aacd(m)акс + aconcd(x) = aconcd(x) \).

For all \( x \in M \) and \( \alpha \in \Gamma \). That is

\( \Gamma \Gamma d(x) = 0, \text{ for all } x \in M \).

Since \( M \) is prime, either \( a = 0 \) or \( d \) is zero.

**Lemma 3.2**: Let \( M \) be a prime \( \Gamma \)-ring of characteristic not \( 2 \) and \( d_1, d_2 \) be a derivation of \( M \) such that the iterate \( d_1d_2 \) is also a derivation. Then one at least of \( d_1, d_2 \) is zero.

**Proof:**
We have \( d_1d_2 \) is a derivation of \( M \) that is,

\( d_1d_2(acab) = d_1d_2(a)cab + aacd_1(b), \text{ for all } a,b \in M \) and \( \alpha \in \Gamma \).
But \( d_1, d_2 \) are each derivation so,
\[
d_1 d_2 (a c b) = d_1 (d_2 (a c b)) + d_2 (a c d_1 (b)) + d_1 (a c d_2 (b) + a c d_1 (d_2 (b))).
\]

But,
\[
d_1 d_2 (a c b) = d_1 (d_2 (a c b)) + a c d_1 (d_2 (b))
\]

So,
\[
d_2 (a c d_1 (b)) + d_1 (d_1 (a c d_2 (b)) = 0,
\]

for all for all \( a, b, c \in M \) and \( \alpha \in \Gamma \) ...

(1)

Replace \( a \) in the last equation by \( a c d_1 (c) \)
\[
d_2 (a c d_1 (c)) c d_1 (b) + d_1 (a c d_1 (c)) c d_2 (b) = 0, \text{ for all } a, b, c \in M \text{ and } \alpha \in \Gamma .
\]

That is
\[
a c (d_2 (d_1 (c) c d_1 (b))) + d_1 (d_1 (a c d_2 (c)) c d_1 (b)) = 0
\]

for all \( a, b, c \in M \) and \( \alpha \in \Gamma \).

Which is merely equation (1) with a replaced by \( d_1 (c) \), then we are left with
\[
d_2 (a c d_1 (c)) c d_1 (b) + d_1 (a c d_1 (c)) c d_2 (b) = 0, \text{ for all } a, b, c \in M \text{ and } \alpha \in \Gamma .
\]

But,
\[
d_1 (a c d_2 (b)) = -d_1 (a c d_1 (b)) \text{ by replacing } a \text{ by } c \text{ the last equation becomes,}
\]
\[
d_2 (a c d_1 (c)) c d_1 (b) = 0, \text{ for all } a, b, c \in M \text{ and } \alpha \in \Gamma .
\]

Factoring out \( a c d_1 (b) \) on the right, we have
\[
(d_2 (a c d_1 (c)) - d_1 (a c d_2 (c)) c d_1 (b)) = 0,
\]

for all \( a, b, c \in M \) and \( \alpha \in \Gamma .
\]

And by Lemma 3.1 unless \( d_1 = 0 \) we have,
\[
(d_2 (a c d_1 (c)) - d_1 (a c d_2 (c)) c d_1 (b)) = 0,
\]

for all \( a, c \in M \) and \( \alpha \in \Gamma .
\]

Replace \( b \) by \( c \) in (1) then,
\[
(d_2 (a c d_1 (c)) + d_1 (a c d_2 (c))) = 0,
\]

for all \( a, c \in M \) and \( \alpha \in \Gamma .
\]

Adding these last two equations, we get
\[
2d_2 (a c d_1 (c)) = 0, \text{ for all } a, b, c \in M \text{ and } \alpha \in \Gamma .
\]

Since characteristic of \( M \) not equal 2, then
\[
d_2 (a c d_1 (c)) = 0, \text{ or else } d_1 = 0 \text{ using Lemma \( 3.1 \) again with } a \text{ replacing } d_2 (a)
\]

we get, either
\[
d_1 = 0 \text{ or } d_2 = 0
\]

Lemma3.3: Let \( M \) be a prime \( \Gamma \) -ring of characteristic different from 2, \( U \) be Lie ideal of \( M \) and \( d \) be anon zero derivation of \( M \).

Then if \( d \) is \( \Gamma \) -centralizing on \( U \) and \( u c a u \in U \), for all \( u \in U \) and \( \alpha \in \Gamma \), then \( M \) is \( \Gamma \) - commuting on \( U \).

Proof:
We have \( d \) is \( \Gamma \) -centralizing on \( U \)
i.e.
\[
[u, d(u)]_x = Z, \text{ for all } u \in U, \text{ and } \alpha \in \Gamma .
\]

Linearizing the above relation on, \( u = u + u c a \), we get
\[
[u c a, d(u)]_x + [u, u c a d(u) + d(u) c a u]_x \in Z,
\]

for all \( u \in U, \text{ and } \alpha \in \Gamma .
\]

That is,
\[
4[u, d(u)]_x = Z, \text{ for all } u \in U, \text{ and } \alpha \in \Gamma .
\]

Since characteristic of \( M \) not equal 2
\[
[u, d(u)]_x \in Z, \text{ then we get }
\]
\[
[u, d(u)]_x = 0, \text{ for all } m \in M, u \in U \text{ and } \alpha, \beta \in \Gamma .
\]

If for some \( u \in U, \text{ } [u, d(u)]_x \neq 0 \) then we get
\[
[u, m]_{\beta} = 0, \text{ in particular } [u, d(u)]_x = 0
\]

Hence,
\[
[u, d(u)]_x = 0, \text{ for all } u \in U, \text{ and } \alpha \in \Gamma .
\]

Lemma3.4: Let \( M \) be a prime \( \Gamma \) -ring, \( U \) be a Lie ideal of \( M \) and \( d \) a nonzero derivation of \( M \).

If \( d \) is \( \Gamma \) -centralizing on \( U \) then
\[
[[d(m), u]_{\alpha}, u]_x \in Z,
\]

for all \( m \in M, u \in U \) and \( \alpha, \beta \in \Gamma .
\]

Further, if \( d \) is \( \Gamma \) -commuting on \( U \) then,
\[
[[d(m), u]_{\beta}, u]_x = 0,
\]

for all \( m \in M, u \in U \) and \( \alpha, \beta \in \Gamma .
\]

Proof:
Since \( U \) is Lie ideal then,
\[
[u, m]_{\alpha} \in U,
\]

for all \( u \in U, m \in M \) and \( \alpha \in \Gamma .
\]

So that, \([u + [u, m]_{\beta}, d(u + [u, m]_{\beta})]_x \in Z.
\]

That is,
\[
[[u, m]_{\beta}, d(u)]_x + [u, [d(u), m]_{\beta}]_x + [u, [d(u), m]_{\beta}]_x \in Z,
\]

for all \( m \in M, u \in U \) and \( \alpha, \beta \in \Gamma .
\]

Now since, for any for all \( m \in M, u \in U \) and \( \alpha, \beta \in \Gamma \) and by (*) we have
\[
[[u, m]_{\beta}, d(u)]_x + [u, [d(u), m]_{\beta}]_x = [m, [d(u), u]_{\beta}]_x \in Z.
\]

By \( \Gamma \) -centralizing of \( d \) we get,
\[
[[u, m]_{\beta}, d(u)]_x + [u, [d(u), m]_{\beta}]_x = 0.
\]

Hence,
\[[d(m),u]_{\alpha},u\]_{\alpha} \in Z,
for all \(m \in M, u \in U\) and \(\alpha, \beta, \in \Gamma\).
The last part can be obtained similarly.

**Lemma 3.5:** Let \(M\) be a prime \(\Gamma\)-ring of characteristic not equal to 2 and 3, and let \(U\) be a Lie ideal of \(M\), if \(d\) is \(\Gamma\)-centralizing on \(U\) then \(d\) is \(\Gamma\)-commuting on \(U\).

**Proof:**
Since \(d\) is \(\Gamma\)-centralizing then, by Lemma 3.4, we have
\[[d(m),u]_{\alpha},u\]_{\alpha} \in Z,
for all \(m \in M, u \in U\) and \(\alpha, \beta, \in \Gamma\).
By using the assumption (*) we get
\[u\beta u\alpha cd(m) + d(m)\alpha u\beta \beta u\beta - 2m\beta d(m)\alpha u\beta \in Z,
for all \(m \in M, u \in U\) and \(\alpha, \beta, \in \Gamma\).
(2)
Commuting with \(u\), we have
\[3m\beta u\alpha cd(u) + u\beta u\alpha cd(m) = 3m\beta d(m)\alpha u\beta + d(m)\alpha u\beta \beta \]
In (3) replace \(m\) by \(u\) and using \(d\) is \(\Gamma\)-centralizing,
\[u\beta u\alpha cd(u) - d(u)\alpha u\beta \beta \]
Furthermore,
\[2(u\beta u\alpha cd(u) - d(u)\alpha u\beta \beta) = u\beta u\alpha cd(u) - d(u)\alpha u\beta \beta,
\]
Write \(d(m) = m'\) and then by replacing \(m\) by \(u\alpha m'\) in (4), we get
\[3u\beta u\alpha cd(m') + u\beta u\alpha cd(m') - u\alpha cd(m')\alpha u\beta + 3m\beta d(u)\alpha m' u\beta + u\beta u\alpha cd(u) - 3m\beta u\alpha cd(u) - d(u)\alpha m' u\beta = 0,
for all \(m \in M, u \in U\) and \(\alpha, \beta, \in \Gamma\).
(6)
However, by assumption (*) and (4), we have
\[3u\beta u\alpha cd(m') + u\beta u\alpha cd(m') - u\alpha cd(m')\alpha u\beta = 3u\beta u\alpha cd(m') + u\beta u\alpha cd(m') - u\alpha cd(m')\alpha u\beta = 0.
Then equation (6) becomes,
\[3u\beta u\alpha cd(u) + u\beta u\alpha cd(u) - d(u)\alpha m' u\beta + u\beta u\alpha cd(u) - d(u)\alpha u\beta \beta = 0.
Using (5) and (6), we arrive at after dividing by 3,
\[(u\alpha u - d(u)\alpha u)\alpha (m'\alpha + m\alpha + u\beta u\beta - 2u\beta u\beta) = 0,
for all \(m \in M, u \in U\) and \(\alpha, \beta, m \in \Gamma\).
If \(u\beta (m'\alpha + u\beta u\beta) - 2u\beta u\beta) = 0\), for some \(u \in U\) and \(\alpha, m \in \Gamma\).
Then we have
\[m'\alpha + u\beta u\beta - 2u\beta u\beta = 0\),
Replace \(m\) by \(u\beta m\) in (9) and using (*) we get,
\[u\beta m'\alpha + u\beta u\beta u\beta - 2u\beta u\beta u\beta + d(u)\beta m\alpha u\beta + u\beta u\beta (u\beta m)\beta - 2u\beta d(u)\alpha m\beta = 0\)
By using (9) we get,
\[u\beta (m'\alpha + u\beta u\beta) - 2u\beta u\beta) = 0\),
Then equation (10) becomes,
\[d(u)\beta m\alpha u\beta + u\beta u\beta (u\beta m)\beta - 2u\beta d(u)\alpha m\beta = 0\)
Now in (9) replace \(m\) by \(u\), and multiply this on the right by \(\beta m\),
\[d(u)\beta m\alpha u\beta + u\beta u\beta (u\beta m)\beta - 2u\beta d(u)\alpha m\beta = 0\)
Subtract (12) from (11),
\[d(u)\beta (m\alpha u\beta - u\beta m) - 2u\beta d(u)\alpha m\beta = 0\)
Multiply (13) by \(u\beta m\) from left and then subtract the results from (14),
\[(u\beta d(u) - m\beta m)\beta (m\alpha u\beta - u\beta m) - 2u\beta (u\beta d(u) - d(u)\alpha m)\beta (m\alpha u\beta - m\beta u\beta) = 0.
Since, \(u\alpha d(u) - d(u)\alpha u\beta = 0\), for all \(u \in U\)
and \(\alpha \in \Gamma\).
Then,
\[m\alpha u\beta - u\beta m\beta - 2u\beta m\alpha - u\beta m = 0\),
for all \(m \in M\).
So, \(m\alpha u\beta - u\beta m\beta - 2u\beta m\alpha - u\beta m = 0\), that is \(u\beta m\alpha - u\beta m\beta = 0\),
That is \(u\) is the center by Lemma 3.2 or else \(u\alpha d(u) - d(u)\alpha u\beta = 0\),
Which in both cases
\[[u, d(u)]_{\alpha} = 0\] for all \(u \in U\) and \(\alpha \in \Gamma\).
The following lemma may have some independent interest.
**Lemma 3.6**: Let $M$ be a prime $\Gamma$-ring of characteristic not 2, $U$ be Jordan ideal of $M$ and $d$ be a nonzero derivation of $M$. If $u\alpha d(u) = d(u)\alpha u = 0$, for all $u \in U, \alpha \in \Gamma$. Then $U = 0$.

**Proof:**
Linearizing the relation $u\alpha d(u) = 0$ on $u = u + w$ where $w \in U$ to get,

$$u\alpha d(w) + w\alpha d(u) = 0,$$
for all $u, w \in U$ and $\alpha \in \Gamma$. ... (15)

For $u \in U$ and any $m \in M, \alpha \in \Gamma$,

$$u\alpha (u\alpha m - m\alpha u) + (u\alpha m - m\alpha u)\alpha u \in U.$$

But, $2(m\alpha u\alpha u - \alpha \alpha u\alpha m) =$

$$\{u\alpha (m\alpha u - m\alpha u) + (m\alpha u - m\alpha u)\alpha u\} - \{m\alpha (u\alpha m - \alpha \alpha m) + \alpha \alpha (m\alpha u - m\alpha u)\}$$

As the first and second term on the right hand side are in $U$,

$$2(m\alpha u\alpha u - \alpha \alpha u\alpha m) \in U.$$  

Now since, $2u\alpha \alpha \in U$ and $2(m\alpha u\alpha u - \alpha \alpha u\alpha m) \in U$.

Then, $4u\alpha u\alpha m$ and $4\alpha u\alpha u\alpha m$ are in $U$.

Replacing $w$ by $4u\alpha u\alpha m$ in (15) and using the hypothesis, we get

$$u\alpha d(m)\alpha u\alpha m = 0,$$
for all $m \in M, u \in U$ and $\alpha \in \Gamma$. ... (16)

Replace $w$ by $m\beta u + u\beta m$ and using the hypothesis, we get

$$u\beta d(m)\beta u + u\alpha m\alpha m + u\beta m\beta m = 0,$$
for all $m \in M, u \in U$ and $\alpha, \beta \in \Gamma$.

Multiply by $\alpha u$ on the right and using the assumption (*) together with equation (16) we obtain

$$u\beta d(m)\beta u\alpha m = 0,$$
for all $m \in M, u \in U$ and $\alpha, \beta \in \Gamma$. ... (17)

Again replace $w$ by $4u\alpha u\alpha m$ in (15), we get

$$u\alpha u\alpha d(m) = 0,$$
for all $m \in M, u \in U$ and $\alpha \in \Gamma$.

Then by Lemma 3.1, we have

$$u\alpha u\alpha = 0,$$
for all $u \in U$ and $\alpha \in \Gamma$.

For $m \in M, u \in U$ and $\alpha \in \Gamma$,

$$2(u\alpha u\alpha m + m\alpha u\alpha u) \in U.$$  

That is,

$$2^3[(u\alpha u\alpha m + m\alpha u\alpha u)\alpha] \in (u\alpha u\alpha m + m\alpha u\alpha u) = 0,$$
for all $m \in M, u \in U$ and $\alpha \in \Gamma$.

Multiply from the right side by $u\alpha u\alpha m$ we get

$$2^3[(u\alpha u\alpha m)\alpha] \in 0,$$
for all $m \in M, u \in U$ and $\alpha \in \Gamma$.

If for some $u \in U$ and $\alpha \in \Gamma$, $u\alpha \neq 0$ then $u\alpha u\alpha M$ is a nonzero right ideal of $M$. Then by Levitzki’s Theorem [13] $M$ would have a nilpotent ideal; which is impossible for prime $\Gamma$-ring, hence

$$u\alpha = 0,$$
for all $u \in U$ and $\alpha \in \Gamma$.

By repeating the above argument we can show that $u = 0$, for all $u \in U$.

**4. The Main Theorems**

**Theorem 4.1**: Let $M$ be a prime $\Gamma$-ring of characteristic different from 2 and 3. Let $d$ be a nonzero derivation of $M$ and $U$ be a Lie ideal of $M$. If $d$ is $\Gamma$-centralizing on $U$ then $U \subset Z$.

**Proof:**
Since $d$ is $\Gamma$-centralizing on $U$, then by using Lemma 3.5, we have

$$[u, d(u)]_\alpha = 0,$$
for all $u \in U$ and $\alpha \in \Gamma$.

Then by Lemma 3.4, we get

$$[d(m), u]_\beta , u]_\alpha = 0,$$
for all $m \in M, u \in U$ and $\alpha \in \Gamma$. ... (1)

In (1) replace $u$ by $u + w$ where $w \in U$,

$$[d(m), w]_\alpha + [d(m), w]_\alpha u]_\alpha = 0,$$
for all $m \in M, u \in U$ and $\beta \in \Gamma$. ... (2)

Suppose now, $u, w \in U$ are such that $w \alpha v$. Then by replacing $w$ by $w\alpha v$ in (2) we get after using (**),

$$w\alpha [d(m), w]_\beta , v]_\alpha + [d(m), w]_\beta , w]_\alpha v + [d(m), w]_\beta , u]_\alpha v + w\alpha [d(m), v]_\beta , u]_\alpha + [w, u]_\alpha \alpha [d(m), v]_\beta = 0.$$  

In view of (2) the last equation reduces to,

$$[d(m), w]_\alpha u]_\alpha + [w, u]_\alpha \alpha [d(m), v]_\beta = 0.$$  

Replace $v$ by $[t, w]_\beta$ where $t \in M$ in above equation, we have

$$[d(m), w]_\beta , v]_\alpha + [w, u]_\alpha v = 0,$$
for all $t, m \in M, u \in U$ and $\alpha, \beta \in \Gamma$.

Putting $u = w$ in (3), we have

$$[d(m), w]_\beta , [w, w]_\alpha = 0.$$  

(4)

Replace $t$ by $t\alpha d(a)$ in (4) where $a \in M$ yields on expansion and (**).
\[ [d(m), w], \beta \alpha [2[t, w], \alpha [d(a), w] + \]
\[ [t, w], \alpha \alpha (d(a)) + \alpha [d(a), w], w] = 0. \]
By (4) the second term is zero, while by (1) the third term is zero. Hence
\[ [d(m), w], \beta \alpha [t, w], \alpha [d(a), w] = 0, \]
for all \( m, t, a \in M, w \in U \) and \( \alpha \in \Gamma \). \hspace{1cm} ... (5)
Put \( u = [t, w] \) in (3), and linearization it s on \( t = t + d(a) \) where \( a \in M \) together with (1) yields
\[ [t, w], w, \alpha [d(a), w], d(m)] = 0, \]
for all \( m, t, a \in M, w \in U \) and \( \alpha \in \Gamma \). \hspace{1cm} ... (6)
Replace \( t \) by \( d(t) \alpha \) where \( p \in M \) in (6) then by expanding we get,
\[ [d[t], w], \alpha [p, w], + d(t) \alpha \]
\[ [p, w], w], + [d[t], w], w], \alpha \gamma \]
\[ + [d(a), w], \beta = 0. \]
By (6) the second term is zero, while by (1) the third term is zero. Hence
\[ [d(t), w], \alpha [p, w], \gamma [d(a), w], \beta = 0. \]
In view of (5), the last equation reduces to,
\[ [d(t), w], \alpha [p, w], \gamma [d(a), d(m)] = 0, \]
for all \( p, a \in M, w \in U \) and \( \alpha, \gamma \in \Gamma \).
In (5) replace \( t \) by \( t \alpha(a) \) where \( p \in M \) then by using the last equation, we get
\[ [d(m), w], \Gamma M[t, d(p), w], [d(a), w] = 0, \]
for all \( m, a \in M, w \in U \) and \( \alpha, \beta \in \Gamma \).
Since \( M \) is prime either \( [d(m), w], \beta = 0 \) or
\[ [d(p), w], \alpha [d(a), w] = 0. \]
If for all \( m \in M, w \in U \) and \( \beta \in \Gamma \),
\[ [d(m), w], \beta = 0. \]
That is, \( I \Gamma (d(m)) = 0. \)
Then by Lemma 3.1, \( w \in Z \), for all \( w \in U \).
Thus assume there exists a \( w \in U \) such that for some \( m \in M, [d(m), w] \neq 0. \)
That is \( w \notin Z \). Then for all \( a, p \in M, \)
\[ [d(p), w], \alpha [d(a), w] = 0. \]
Replace \( a \) by \( b \beta c \) where \( b, c \in M \) then by expanding, we get
\[ [d(p), w], \alpha [d(b), w], \beta c + [d(p), w] \alpha \]
\[ [d(b), \beta c] + [d(p), w], \alpha [d(c), w] + \]
\[ [d(p), w], \alpha [b, w], \beta d(c) = 0. \]
Replace \( b \) by \( [t, w] \) where \( t \in M \). Then by
(7) the first term is zero, by (5) the third term is zero and by (4) the fourth term is zero, thus
\[ [d(p), w], \alpha [d(t), w] \beta [w, c] = 0. \]
Since, \( d([t, w]) = [d(t), w] + [t, d(w)] \), and using (3), we get
\[ [d(p), w], \alpha [t, d(w)] \beta [w, c] = 0, \]
for all \( c, t, p \in M, w \in U \) and \( \alpha, \beta \in \Gamma \).
Replace \( c \) by \( mac \) where \( m \in M \), then
\[ [d(p), w], \alpha [t, d(w)] \Gamma M[t, w] = 0. \]
Since \( M \) is prime and \( w \notin Z \), we get
\[ [d(p), w], \alpha [t, d(w)] = 0, \]
for all \( t, p \in M, w \in U \) and \( \alpha \in \Gamma \). Thus
\[ [d(p), w], \alpha [t, d(w)] = 0, \]
for all \( t, p \in M, w \in U \) and \( \alpha \in \Gamma \). Which in both cases \( d(w) \notin Z \).
Now suppose that \( u \in U \) and \( u \in Z \) then
\[ 0 = d([u, a]) = [d(u), a] + [u, d(a)] \]
and hence \( d(u) \in Z \). Therefore, \( d(u) \in Z \) for all \( u \in U \). So that,
\[ [w, d(a)] \in Z \]
for all \( a \in M \) that is
\[ [w, d(a)] \in Z. \]
In particular,
\[ [w, d(a)] \in \Gamma \in Z, \]
\[ \beta d(w) \in Z. \]
By commuting (6) with \( w \), we get
\[ [w, [w, a]] \beta d(w) = 0, \]
for all \( a \in M, w \in U \) and \( \alpha, \beta \in \Gamma \).
If \( d(w) \neq 0 \) and as its in the center \( Z \),
\[ [w, [w, a]] \alpha = 0, \]
for all \( a \in M \) and \( \alpha \in \Gamma \).
By sub- Lemma \[ 14 \] \( w \in Z \) a contradiction. Hence, \( d(w) = 0 \). Thus by (8), we have
\[ [w, d(a)] \beta \in Z, \]
for all \( a \in M \) and \( \alpha \in \Gamma \).
That is, \( [w, d(a)] \beta w \in Z, \)
for all \( a \in M \) and \( \alpha \in \Gamma \).
Replace \( b \) by \( c a b \) where \( c \in M \), then
\[ [d(a), w], \beta [w, b] = 0. \]
By primness of \( M \) we get, either \( w \in Z \) or \( [d(a), w] = 0, \) for all \( a \in M \) and \( \alpha \in \Gamma \).
Which we both cases a contradiction Hence, \( w \in Z \) for all \( w \in U \).
Now we should like to settle the problem when \( M \) has characteristic 3. Hence we get the following result.
**Theorem 4.2:** Let $M$ be a prime $\Gamma$-ring of characteristic 3, and $d$ be a nonzero derivation of $M$, if $d$ is $\Gamma$-centralizing on $U$ and $ucau \in U$ then $U \subset Z$.

**Proof:**

Since $d$ is $\Gamma$-centralizing on $U$ then,

By Lemma 3.3 we get $d$ is $\Gamma$-commuting on $U$. Therefore, by similar way of the proof in Theorem 4.1 we can get $U \subset Z$.

Now we show that the conclusion of Theorem 4.1 and Theorem 4.2 holds even if $U$ is Jordan ideal of $M$.

**Theorem 4.3:** Let $M$ be a prime $\Gamma$-ring of characteristic not 2. Let $d$ be a nonzero derivation of $M$ and $U$ be a Jordan ideal of $M$ if $d$ is $\Gamma$-centralizing then $U \subset Z$.

**Proof:**

Since $2ucau \in U$, then by Lemma 3.3, 

$[u,d(u)]_{\alpha} = 0$, for all $u \in U$ and $\alpha \in \Gamma$.

Linearizing the relation $[u,d(u)]_{\alpha} = 0$, on $u = u+v$ where $v \in U$, we get

$[u,(d(v))]_{\alpha} + [v,d(u)]_{\alpha} = 0$, for all $u,v \in U$ and $\alpha \in \Gamma$. ...(9)

In (9), replace $v$ by $u\beta m + m\beta u$ where $m \in M$ then by expanding, we get

$u\beta[u,d(u)]_{\alpha} + [u,d(m)]_{\alpha} + [u,(d(m))]_{\alpha} + [v,d(u)]_{\alpha} + [v,d(m)]_{\alpha} + [v,(d(m))]_{\alpha} + [m,d(u)]_{\alpha} + [m,d(m)]_{\alpha} + [m,(d(m))]_{\alpha} + [m\beta u]_{\alpha} = 0$. i.e.

$2u\beta mcau(u) - 2d(u)cma\beta u + u\beta u(a)d(m) - d(m)ca\beta u = 0$ ...(10)

Replace $m$ by $\alpha m$ in (10), we get

$d(u)\alpha ca(u)bm\beta u - u\beta u(a)m\beta u = 0$, for all $m \in M, u \in U$ and $\alpha, \beta \in \Gamma$. ...(11)

That is, $d(u)\alpha ca(\alpha m) = 0$, for all $m \in M, u \in U$ and $\alpha, \beta \in \Gamma$.

Hence by Lemma 3.1 we have, either $u\beta u \in Z$ or $d(u) = 0$, for all $u \in U$ and $\alpha, \beta \in \Gamma$.

For $u \in U$ and any $m \in M, \alpha, \beta \in \Gamma$, we have $u\alpha m + m\alpha u \in U$. But,

$4u\alpha m\alpha u = 2(u\alpha (u\alpha m + m\alpha u) + (u\alpha m + m\alpha u)\alpha u) - 2(u\alpha c(u\alpha m + m\alpha c)u\alpha u)$. The first and second term on the right are in $U$ then, $4u\alpha m\alpha u \in U$. Replace $v$ by $u\alpha m\alpha u$ in (9), we get

$ucaucau(u) - d(u)caucau + ucaucau(m) - ucaucau(m)caucau = 0$ ...(12)

Replace $m$ by $\alpha m$ in (12) and then by using (12) we get,

$ucaucau(u)caucau - ucaucau(m)caucau = 0$.

In view of (11) the last equation reduces to

$ucaucau(u)caucau - ucaucau(m)caucau = 0$.

That is, $ucaucau(u)caucau(m) = 0$.

Then by Lemma 3.1, we have either $ucaucau(m) = 0$ or $U \subset Z$, for all $u \in U$ and $\alpha \in \Gamma$.

In (11), replace $u$ by $u+v$ where $v \in U$ then by using (11), we get

$d(u)\alpha ca[v\beta u + v\beta u, m]_{\alpha} + d(u)\alpha ca[v\beta u, m]_{\alpha} + d(u)\alpha ca[v\beta u, m]_{\alpha} = 0$.

Replace $u$ by $-u$ then,

$-d(u)\alpha ca[v\beta u - v\beta u, m]_{\alpha} - d(u)\alpha ca[v\beta u, m]_{\alpha} + d(u)\alpha ca[v\beta u, m]_{\alpha} = 0$.

Adding the last two equations and dividing by 2, we have

$d(u)\alpha ca[v\beta u + v\beta u, m]_{\alpha} + d(u)\alpha ca[v\beta u, m]_{\alpha} = 0$ for all $m \in M, u,v \in U$ and $\alpha, \beta \in \Gamma$.

By Lemma 3.6 we get $ucaucau(m) = 0$, for some $u \in U, \alpha \in \Gamma, d(u) \neq 0$.

Hence by (12), $u\beta u \in Z$. The net results of this is

$d(u)\alpha ca[v\beta u + v\beta u, m]_{\alpha} = 0$,

for all $m \in M, u,v \in U$ and $\alpha, \beta \in \Gamma$.

That is, $d(u)\alpha ca(u)\beta v + u\beta u(a)m\beta u = 0$,

for all $m \in M, u,v \in U$ and $\alpha, \beta \in \Gamma$.

By Lemma 3.1, $v\beta u + v\beta u \in Z$, for all $u,v \in U$ and $\alpha, \beta \in \Gamma$.

If $ucau = 0$, then

$0 = d(ucau) = ucau + d(u)cau = ucau$.

That is, $ucau = 0$ a contradiction hence $ucau \neq 0$. Now suppose that $ucau = 0$, then $ucaucau(u) = 0$ that is, $d(u) = 0$ a contradiction hence $ucaucau(m) = 0$.

So by (13) $U \subset Z$. Hence $2u\alpha v \in Z$; that is

$2u\alpha v \in Z$ for all $v \in U$ and $\alpha \in \Gamma$.

As $u \neq 0$ we have $v \in Z$ for all $v \in U$.

Hence $U \subset Z$.

We should like to settle the problem even when $M$ has characteristic 2. In this case Lie and
Jordan ideals will coincide.

**Theorem 4.4:** Let $M$ be a prime $\Gamma$-ring of characteristic 2, and let $d$ be a nonzero derivation of $M$. Let $U$ be Lie (Jordan) ideal and subring of $M$. If $d$ is $\Gamma$-centralizing on $U$ then $U$ is commutative.

**Proof:**

Since $d$ is $\Gamma$-centralizing on $U$ then by Lemma 3.4

$$d(m)\beta uau + uau\beta d(m) \in Z$$ \hspace{1cm} (14)

Commute (14) with $d(m)$ and $uau$ respectively we get,

$$uau\beta d(m)\gamma d(m)\beta uau = d(m)\gamma d(m)\beta uau$$ \hspace{1cm} (15a)

And,

$$d(m)\beta uau\beta uau = uau\gamma uau\beta d(m)$$ \hspace{1cm} (15b)

in (15a) replace $m$ by $v + uau\beta v$ and by using (15 a) we get,

$$uau\beta d(v + uau\beta v)\gamma d(v + uau\beta v) = d(v + uau\beta v)\gamma d(v + uau\beta v)\beta$$

For $u \in U, \alpha \in \Gamma$,

$$d(uau) = uau d(u) + d(u) uau \in Z.$$ 

So in view of (15b) the last equation reduces to

$$uau\beta d(v)\gamma uau\beta d(v) + d(v)\gamma uau\beta d(v)\beta$$

$uau = 0$, for all $u, v \in U, \alpha \in \Gamma$.

Since $M$ is prime, and by using (14) we get,

$$uau\beta d(v) = d(v)\beta uau, \text{ for all } u, v \in U, \alpha \in \Gamma.$$ \hspace{1cm} (16)

Replace $u$ by $u + w$ where $w \in U$ we get,

$$(uau + wau)\beta d(v) = d(v)\beta (uau + wau)$$

Replace $v$ by $vau$ and by using (*) we have,

$$(uau + wau)\beta (uau + d(v)au) = 0,$$

for all $u, v, w \in U, \alpha, \beta \in \Gamma$. \hspace{1cm} (17)

Linearize the last equation on $u = u + vau$ where $v \in U$ and put $v = u$ then using (16) we get,

$$(vau + wau)\beta (uau + d(u)au) = 0,$$

for all $u, v, w \in U, \alpha, \beta \in \Gamma$.

If $[u, d(u)]_{\alpha} \neq 0$, for some $u \in U$ and $\alpha \in \Gamma$. Then,

$$(vau + wau)\alpha = 0,$$

for all $v, w \in U$ and $\alpha \in \Gamma$. So that,

$$uau(uau + maau) = (uau + maau)uau$$

That is

$$w\alpha(uau + maau) = (uau + maau)\alpha.$$\hspace{1cm} Replace $m$ by $mau$ then

$$(uau + maau)\alpha(wau + uau) = 0,$$

for all $m \in M, u, w \in U$ and $\alpha \in \Gamma$.

Replace $w$ by $[u, t]_{\alpha}$ we get,

$$(uau + maau)\alpha(uau + tmaau) = 0,$$

for all $m, t \in M, u, w \in U$ and $\alpha \in \Gamma$.

Replace $t$ by $pau$ where $p \in M$, then

So that,

$$uau(uau + maau) = (uau + maau)uau$$

That is

$$w\alpha(uau + maau) = (uau + maau)\alpha.$$\hspace{1cm} Replace $m$ by $mau$ then

$$(uau + maau)\alpha(wau + uau) = 0,$$

for all $m \in M, u, w \in U$ and $\alpha \in \Gamma$.

Replace $w$ by $[u, t]_{\alpha}$ we get,

$$(uau + maau)\alpha(uau + tmaau) = 0,$$

for all $m, t \in M, u, w \in U$ and $\alpha \in \Gamma$.
By primness of $M$ we have, $w\alpha v \in Z$ a contradiction. Hence the conclusion is that, so in all possible cases, $w\alpha v \in Z$, for all $u \in U, \alpha \in \Gamma$. So that, 

$(u\alpha v + v\alpha u) \in Z$ and $(u\alpha v + v\alpha u)\alpha u \in Z$

If $u \not\in Z(U)$ where $Z(U)$ denotes the center of, then $(u\alpha v + v\alpha u = 0$, for all $v \in U$ and $u \in Z(U)$

Hence $U$ is commutative.

References
