ON CLS-MODULES

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Abstract
Let R be a commutative ring with identity and let M be a unital left R-module. A. Tercan introduced the following concept. An R-module M is called a CLS-module if every y-closed submodule is a direct summand. The main purpose of this work is to develop the properties of y-closed submodules.

Keywords: CLS-modules, Y-closed submodules.

1. Introduction
About thirty years ago, M. Harada and B. Muller introduced the following concept. An R-module M is called extending (briefly CS-module) if every submodule is essential in a direct summand of M, see[1]. Equivalently, M is an extending module if and only if every closed submodule of M is a direct summand. Extending modules has been studied recently by several authors. Among them P.F. Smith and Clark and Mohamed see [1, 2].

Now recall that, a submodule N of an R-module M is a y-closed submodule if $\frac{M}{N}$ is non-singular, see[3]. It is easily seen that every y-closed is closed.

A. Tercan generalizes the extending modules as follow: An R-module M is called a CLS-module if every y-closed submodule of M is a direct summand, see[4]. Note that, Tercan used the concepts complement(closed submodule) in the sense of closed submodule (y-closed submodule). CLS-modules also have been studied by Yongduo Wang, see [5].

In this paper, we give some results on y-closed submodules and CLS-modules.

In section one, we study the properties of y-closed submodules. We prove that if $f:M \rightarrow N$ is an epimorphism and $B \subseteq N$, then for every singular submodule A of M, $f(A) \subseteq B$, see proposition 1.4.

In section two of the paper, we give characterizations of CLS-modules. For example, we show that an R-module M is CLS if and only
if for every y-closed submodule A of M, there is a decomposition M=M₁ ⊕ M₂ such that A⊆ M₁ and M₂ is a complement of A in M.

Y-closed Submodule

Proposition(1.1) :
Let M be an R-module and let A⊆ B⊆ M ,then
1- If A⊆ M, then A⊆ y B.
2- Let A ⊆ B ⊆ M ,then B ⊆ y M if and only if
   \( \frac{B}{A} \subseteq y \frac{M}{A} \).
3- A ∩ B ⊆ y B if and only if
   A ∩ B ⊆ y B.
4- If A ∩ y M and B ∩ y M, then
   A ∩ B ⊆ y M.

Proof:
1- Clear.
2- Clear by the third isomorphism theorem.
3- Clear by the second isomorphism theorem.
4- Assume that A ⊆ y M and B ⊆ y M to show that A∩B ⊆ y M ,let m ∈ M such that
   \( m+(A ∩ B) ∈ Z(\frac{M}{A ∩ B}) \).
   Thus ann(m+A ∩ B) ⊆ R. Since
   \( \text{ann}(m+A ∩ B) \subseteq \text{ann}(m+A) \), then
   \( \text{ann}(m+A) ⊆ R \), by [3]. But \( Z(\frac{M}{A}) = 0 \), therefore \( m+\text{ann}(m+A) = A \).
   By the same way \( m+B = B \), so
   \( m ∈ A ∩ B \) and hence \( Z(\frac{M}{A ∩ B}) = 0 \).

Proposition(1.2) :
Let A and B be a submodule of an R-module M if A ⊆ y B and B ⊆ y M , then A ⊆ y M.

Proof:
Let A ⊆ y B and B ⊆ y M. Now consider the following short exact sequence:

\[
0 \rightarrow \frac{B}{A} \rightarrow \frac{M}{A} \rightarrow \frac{A}{B} \rightarrow 0
\]

Where i is the inclusion map and π is the natural epimorphism. Since A ⊆ B and

B ⊆ M, then \( \frac{B}{A} \subseteq \frac{M}{A} \) by proposition 1.1-2.

Since \( \frac{B}{A} \) and \( \frac{M}{A} \) are non-singular, then \( \frac{M}{A} \) is non-singular by [3].

Note:
Let M be an R-module and A ⊆ B ⊆ M, then
1- If B ⊆ y M, then A need not be y-closed submodule of M, for example:
   Consider Z as Z-module and
   \( 2Z ⊆ Z \) it is clear that Z ⊆ y Z. But
   \( Z(\frac{Z}{2Z}) = Z(Z_2) = Z_2 \) is singular.
2- If A ⊆ y M, then B need not be y-closed in M, for example:
   \( 0 ⊆ 2Z \subseteq Z \) Clearly 0 ⊆ y Z, but
   \( Z(\frac{Z}{2Z}) = Z(Z_2) = Z_2 \) is singular.

Note:
An epimorphic image of an y-closed submodule need not be y-closed submodule as the following example show:-

Let \( \pi : Z \rightarrow \frac{Z}{4Z} \) be the natural epimorphism
   Clearly 0 ⊆ y Z, but \( f(0) = 0 \) is not y-closed in
   \( \frac{Z}{4Z} \) (because \( \frac{Z}{4Z} \) is singular).

Proposition(1.3) :
Let f:M → N be an epimorphism and A ⊆ y M. If Ker f ⊆ A, then f(A) ⊆ y N.

Proof:
Assume that A ⊆ y M. To show that
f(A) ⊆ y N, let n → N such that
\( \text{ann}(n+f(A)) \subseteq R \). Since f is an epimorphism, then \( n = f(m) \), for some m ∈ M. Since
Ker f ⊆ A, then \( \text{ann}(n+f(A)) \subseteq \text{ann}(m+A) \) and hence \( \text{ann}(n+f(A)) ⊆ R \), by [3]. But A ⊆ y M, therefore m ∈ A. Thus n = f(m) ∈ f(A).

Proposition(1.4):
Let f:M → N be an R-homomorphism and B ⊆ y N, then for every singular submodule A of M, f(A) ⊆ B.
Proof:
Let \( \pi : N \to \frac{N}{B} \) be the natural epimorphism.

Consider \( \pi \circ f : M \to \frac{N}{B} \).

Now \( \pi \circ f |_A : A \to \frac{N}{B} \)

But \( A \) is singular and \( \frac{N}{B} \) is non-singular,

Therefore \( \pi \circ f |_A = 0 \), by [3]. Thus \( \pi(f(A)) = 0 \)

And hence \( f(A) \subseteq \ker \pi = B \).

The following corollary follows immediately from proposition\{ 1.4 \}.

**Corollary (1.5):**
Let \( N \) be an \( R \)-module and \( A \subseteq \frac{1}{y}N \). Then \( (\text{HOM}(M,N))(M) \subseteq B \), for every singular \( R \)-module \( M \).

**Proposition (1.6):**
Let \( M \) be an \( R \)-module and \( A \subseteq \frac{1}{y}M \), then \( Z(M) = Z(A) \).

**Proof:**
It is enough to show that \( Z(M) \subseteq Z(A) \)

Let \( i : Z(M) \to M \) be the inclusion map and

\( \pi : M \to \frac{M}{A} \) be the natural epimorphism.

Consider the map \( \pi \circ i : Z(M) \to \frac{M}{A} \)

Since \( Z(M) \) is singular and \( \frac{M}{A} \) is non-singular, then \( \pi \circ i = 0 \), by [3]. So \( \pi \circ i : (Z(M)) = \pi \circ (Z(M)) = 0 \).

Thus \( Z(M) \subseteq \ker \pi = A \). But \( Z(A) = Z(M) \cap A \),

Therefore \( Z(A) = Z(M) \).

**Proposition (1.7):**
Let \( M \) be an \( R \)-module and let \( A \subseteq B \subseteq M \)

and \( A \subseteq \frac{1}{y}M \), then \( \frac{M}{B} \) is singular if and only \( B \subseteq e \frac{M}{e} \).

**Proposition (1.8):**
Let \( M \) be an \( R \)-module and \( B \) be a maximal and \( y \)-closed submodule of \( M \). Then \( \frac{M}{B} \) is projective and \( B \) is a direct summand of \( M \).

**Proof:**
Since \( B \) is maximal submodule of \( M \), then \( \frac{M}{B} \) is simple and hence semisimple. But \( \frac{M}{B} \) is non-singular, therefore \( \frac{M}{B} \) is projective, by [3].

Now consider the following short exact sequence:

\[
0 \to B \xrightarrow{i} M \xrightarrow{\pi} \frac{M}{B} \to 0
\]

Where \( i \) is the inclusion map and \( \pi \) is the natural epimorphism.

Since \( \frac{M}{B} \) is projective, then the sequence is splits, by [6]. Thus \( B \) is a direct summand of \( M \).

Let \( M \) be an \( R \)-module and \( N \subseteq M \). Recall that the residual of \( M \) in \( N \) (denoted by \([N:M]\)) is defined as follows:

\[
[N:M] = \{ r \in R : rM \subseteq N \}, \text{ see } [7]
\]

**Proposition (1.9):**
Let \( M \) be an \( R \)-module and \( N \subseteq \frac{1}{y}M \), then \([N:M] \subseteq \frac{1}{y}R \).
Proof:
Let \( N \subseteq M \). Assume that \([N:M]\) is not y-closed in \( R \). So there exists \( r \in R \) such that
\[ [N:M] \neq r + [N:M] \in \mathbb{Z}\left( \frac{N}{[N:M]} \right) . \]
Thus \( rM \not\subseteq N \) and hence \( \exists m_0 \in M \) such that \( rm_0 \not\in N \). One can easily show that
\[ \text{ann}(r + [N:M]) \subseteq \text{ann}(rm_0 + N) \]. Since
\[ \text{ann}(r + [N:M]) \subseteq R, \text{ then ann}(rm_0 + N) \subseteq R \]
But \( M \) is non-singular, therefore \( m_0 + N = N \).
Which is contradiction.

Proposition(1.10):
Let \( M \) be an R-module and let \([ B_\alpha, \in \wedge \)] be an independent family of submodules of \( M \) and \( A \subseteq B_\alpha \), \( \forall \alpha \in \wedge \). Then \( \bigoplus A_\alpha \subseteq \bigoplus B_\alpha \) if and only if \( A_\alpha \subseteq B_\alpha \), \( \forall \alpha \in \wedge \).

Proof:
Suppose that \( \bigoplus A_\alpha \subseteq \bigoplus B_\alpha \)
\[ \bigoplus B_\alpha \cong B_\alpha \]
Then by \([3]\)
\[ A_\alpha \subseteq B_\alpha, \forall \alpha \in \wedge \].
Conversely, \( A_\alpha \subseteq B_\alpha, \forall \alpha \in \wedge \).
Then \( B_\alpha \) is non-singular, \( \forall \alpha \in \wedge \).
and hence
\[ \bigoplus B_\alpha \]
\[ \bigoplus A_\alpha \]
\[ \subseteq \bigoplus B_\alpha \]
\[ \subseteq \bigoplus A_\alpha \]
But \( \bigoplus A_\alpha \cong \bigoplus B_\alpha \), by \([8]\), so
\[ \bigoplus A_\alpha \subseteq \bigoplus B_\alpha \] .

1. Characterizations of CLS-modules
   Following \([4]\), we say that an R-module \( M \) is a CLS-module if every y-closed submodule is a direct summand.
   It is know that a direct summand of a CLS-module is CLS, see \([4]\).
   We prove the following:

Proposition(2.1):
Every y-closed submodule of CLS-module is CLS.

Proof:
Let \( M \) be a CLS-module and \( A \subseteq M \). We want to show that \( A \) is a CLS-module. Let \( K \subseteq A \), then by proposition \([1.2]\) \( K \subseteq M \). But \( M \) is CLS, therefore \( K \) is a direct summand of \( M \) and hence \( K \) is a direct summand of \( A \).

Proposition(2.2):
Let \( M \) be a CLS-module and \( N \) be a submodule of \( M \), then \( \frac{M}{N} \) is a CLS-module.

Proof:
Let \( \frac{B}{N} \subseteq \frac{M}{N} \). Then by proposition \([1.1-2]\) \( B \subseteq M \). But \( M \) is CLS, therefore
\( M = B \oplus K, K \subseteq M \). Since \( N \subseteq B \), then one can easily show that \( \frac{M}{N} = \frac{B}{N} \oplus \frac{K + N}{N} \). Thus \( \frac{M}{N} \) is CLS-module.
Recall that a module \( M \) is called generalized extending if for any submodule \( N \) of \( M \), there is a direct summand \( K \) of \( M \) such that \( N \subseteq K \) and
\[ \frac{K}{N} \]
is singular, see \([9]\).

Proposition(2.3):
Let \( M \) be a generalized extending R-module, then \( M \) is CLS.

Proof:
Let \( N \subseteq M \). Since \( M \) is generalized extending, then there exists a direct summand \( K \) of \( M \) such that \( N \subseteq K \) and
\[ \frac{K}{N} \]
is singular. But
\[ \frac{K}{N} \subseteq \frac{M}{N} \]
is non-singular. Thus \( K = N \).

Proposition(2.4):
An R-module \( M \) is a CLS-module if and only if for every y-closed submodule \( A \) of \( M \), there is a decomposition \( M = M_1 \oplus M_2 \) such that \( A \subseteq M_1 \) and \( M_2 \) is a complement of \( A \) in \( M \).

Proof:
Let \( A \subseteq M \) then by our assumption, there exists a decomposition of \( M \) into two submodules \( A \subseteq M_1 \) and \( M_2 \) is a complement of \( A \) in \( M \). So
\[ A \oplus M_2 \subseteq M \], by \([3]\). Thus \( A \subseteq M_1 \) by \([3]\) and
\[ M_1 \]
\[ \subseteq \frac{M}{N} \]
is singular. But \( A \subseteq M_1 \) and
\[ A \subseteq M_1 \]
therefore \( A \subseteq M_1 \), by Proposition \([1.1-1]\) . Thus \( A = M_1 \).

Proposition(2.5):
An R-module \( M \) is CLS-module if and only if every y-closed submodule of \( M \) is essential in a direct summand.

Proof:
\( \Rightarrow \) Clear.
Since $R$ is projective, then $Ra$ is a direct summand of $M$. Thus $\frac{D}{A}$ is singular.

But $\frac{D}{A} \subseteq \frac{M}{A}$, therefore $\frac{D}{A}$ is non-singular. Thus $A=D$ and hence $M$ is CLS.

**Proposition (2.6):**
An $R$-module $M$ is CLS-module if and only if for every $\gamma$-closed submodule $A$ of $M$ there exists a decomposition $M=M_1 \oplus M_2$ such that $A \subseteq M_1$ and $A \oplus M_2 \subseteq M$.

**Proof:**
\[ \Rightarrow \) Clear.
\[ \Leftarrow \) Let $A \subseteq \gamma M$, we want to show that $A$ is a direct summand of $M$. Since $A \subseteq \gamma M$, then by assumption there exists a decomposition $M=M_1 \oplus M_2$ such that $A \subseteq M_1$ and $A \oplus M_2 \subseteq M$.

So $\frac{M}{A \oplus M_2}$ is singular by [3]. But $A \subseteq M_1$ and $A \subseteq M_1 \oplus M_2$, therefore by proposition (1.1-1) $A \subseteq M_1$. Since $M_2 \subseteq M_2$, then by proposition (1.10) $A \oplus M_2 \subseteq M_1 \oplus M_2 = M$. So $\frac{M}{A \oplus M_2}$ is singular. Thus $M = A \oplus M_2$.

**Proposition (2.7):**
An $R$-module $M$ is CLS-module if and only if for every direct summand $A$ of the injective hull $\text{E}(M)$ of $M$ such that $A \bigcap M \subseteq \gamma M$, then $A \bigcap M$ is a direct summand of $M$.

**Proof:**
\[ \Rightarrow \) Clear.
\[ \Leftarrow \) Let $A \subseteq \gamma M$ and let $B$ be a relative complement of $A$, then by [3]

$A \oplus B \subseteq \gamma M$. Since $M \subseteq \gamma E(M)$, then $A \oplus B \subseteq \gamma E(M)$. Thus $E(A) \oplus E(B) = E(A) \oplus E(B) \subseteq E(M)$. Since $E(A)$ is a summand of $E(M)$, then by our assumption $E(A) \bigcap M$ is a summand of $M$. Now $A \subseteq E(A)$ and $M \subseteq M$, thus by [3]

$A = A \bigcap M \subseteq E(A) \bigcap M$. Hence by proposition (2.5), $M$ is CLS.

**Proposition (2.8):**
Let $R$ be a ring, then $R$ is a CLS-module if and only if every cyclic non-singular $R$-module is projective.

**Proof:**
Let $R$ be a CLS-ring and $M = Ra$, $a \in M$ be a nonsingular $R$-module. Now consider the short exact sequence.

$0 \rightarrow \text{ann}(a) \rightarrow R \xrightarrow{f} Ra \rightarrow 0$

Where $i$ is the inclusion homomorphism and $f$ is a map defined by $f(r) = ra, r \in R$. Clearly that $f$ is an epimorphism and $\ker f = \text{ann}(a)$. Then by first isomorphism theorem, $\frac{R}{\text{ann}(a)} \cong Ra$. But $Ra$ is non-singular, therefore $\text{ann}(a) \subseteq R$.

Since $R$ is CLS, then $\text{ann}(a)$ is a direct summand of $R$, so the sequence is split. Thus by [6] $R \cong \text{ann}(a) \oplus Ra$. Since $R$ is projective, then $Ra$ is projective by [6].

Conversely, let $A$ be a $\gamma$-closed ideal in $R$, then $\frac{R}{A}$ is non-singular. Since $R$ is cyclic, then $\frac{R}{A}$ is cyclic. By our assumption $\frac{R}{A}$ is projective. Now consider the following short exact sequence:

$0 \rightarrow A \xrightarrow{i} R \xrightarrow{\pi} \frac{R}{A} \rightarrow 0$

Where $i$ is the inclusion homomorphism and $\pi$ is the natural epimorphism, since $\frac{R}{A}$ is projective, then the sequence is split by [6]. Thus $A$ is a summand of $R$. It is well known that a direct sum of CLS-modules need not to be a CLS-modules, see [4], so we give some conditions under which this relation is true.

**Proposition (2.9):**
Let $M$ and $N$ be CLS-modules such that $\text{ann}M + \text{ann}N = R$, then $M \oplus N$ is CLS.

**Proof:**
Let $A$ be a $\gamma$-closed submodule of $M \oplus N$. Since $\text{ann}M + \text{ann}N = R$, then by the same way of the prove [11, prop. 4.2, CH. 1], $A = C \oplus D$, where $C$ is a submodule of $M$ and $D$ is a submodule of $N$.

Since $A = C \oplus D \subseteq M \oplus N$, then $C$ and $D$ are $\gamma$-closed submodule in $M$ and $N$ respectively by proposition (1.10).

But $M$ and $N$ are CLS-modules, therefore $C$ is a summand of $M$ and $D$ is a summand of $N$. So $A = C \oplus D$ is a summand of $M \oplus N$. Thus $M \oplus N$ is a CLS-module.
Recall that a submodule N of R-module M is
called a fully invarientsubmodule of M, if for
every endomorphism f:M → M,
\( f(N) \subseteq N \), see[11].

**Proposition(2.10):**
Let M= \( \oplus M_i \) be an R-module ,such that every
y-closed submodule of M is fully invariant ,then
M is CLS if and only if M_i is CLS \( \forall \ i \in I \).
**Proof:**
⇒ Clear .
⇒ ) let S be a y-closed sub module of M .
For each \( i \in I \), let \( \pi_i : M \rightarrow M_i \) be the projection
map .Now Let x ∈ S ,then \( x = \sum_{i \in I} m_i \), \( m_i \in M_i \)
and \( m_i = 0 \) for all but finite many element of \( i \in I \).
\( \pi_i(x) = m_i, \forall i \in I \)
Since S is y-closed submodule, then by our
assumption , S is fully invariant
and hence \( \pi_i(x) = m_i \in S \cap M_i \),So \( x \in \bigoplus (S \cap M_i) \).
Thus \( S \subseteq \bigoplus (S \cap M_i) \).
But \( \bigoplus (S \cap M_i) \subseteq S \),therefore \( S = \bigoplus (S \cap M_i) \).
Since \( S \subseteq M_i \) then by proposition (1.10)
\( (S \cap M_i) \) is a direct summand of \( M_i \)
Thus S is a direct summand on M
Recall that an R-module M is called a
distributive module if
\( A \cap (B+C) = (A \cap B) + (A \cap C) \), for all
submodules A,B and C of M, see[12].

**Proposition(2.11):**
Let \( M = M_1 \oplus M_2 \) be distributive R-
module. Then M is CLS if and only if \( M_1 \)
and \( M_2 \) are CLS.
**Proof:**
⇒ ) Clear. 
⇒ K \( \subseteq \gamma \), M.Since M=M_1 \( \oplus M_2 \), then
K = K \( \cap (M_1 \oplus M_2) \). But M distributive ,
therefore K=(K \( \cap M_1 \) \( \oplus (K \cap M_2) \).by
proposition(1.10) K \( \cap M_1 \subseteq M_1 \) and
K \( \cap M_2 \subseteq M_2 \).
Since \( M_1 \)and \( M_2 \) are CLS, then
(K \( \cap M_1 \) ) is a direct summand of \( M_1 \), and
(K \( \cap M_2 \) ) is a direct summand of \( M_2 \)
Clearly that K=(K \( \cap M_1 \) \( \oplus (K \cap M_2) \) is a direct
summand of M.

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