Development a 3-D Mathematical Model for Network Topology Based on Graph Theory*

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Abstract

Communication network efficiency depends upon many factors among which is the “Topology” which makes topology optimization an important issue to care for. One important thing in optimization problems is the formulation of objective functions. For the case of topology design it is not a straightforward matter to develop an efficient topology model as well as objective functions to be used in the optimization process. An effective unconventional approach is needed. This paper is concerned with enhancing the already existing set of formulas, relating topology and topology properties modeling and topology design objectives, by proposing a 3-dimension way of modeling that can serve network analysis, design, and optimization. The approach is based on graph theory. The proposed model and formulas can be easily programmed.

Keywords: Network topology, Modeling, Graph theory.

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1. Introduction

One of the very important matters in engineering analysis and design is how to model the given problem as all. By the problem it is meant the system under consideration as well as the design targets or objectives. The problem representative model should reflect fairly the features of interest in the given system besides the design objectives, and should be suitable for the following analysis and design work [1] [2]. For this to happen, the representation adopted must suit a sufficient and efficient analysis, design, and optimization methods and tools, whether those already existing in the first place, or those that could be developed.

When optimization methods are judged, their suitability, the time they consume, difficulty, implementation robustness, and accuracy, are among the things to be considered and the objective function is one of the things affecting these points [3]

Different modeling ways do exist among which and the most important one is the mathematical one. The model must be as much as possible direct, simple, general, reasonable mathematically, and practical for computation and programming purposes. Model generality implies that it could serve a lot of work purposes or targets and flexible to cover different features of interest in representation, analysis purposes, and design features and targets [1].

Graph Theory (G.T.) is very useful for the purposes mentioned above when it comes to modeling, analysis, and design of communication network topology; noticing that topology itself is a structure [4].Topology design optimization is a very important issue as it affects cost, survivability, and efficiency of the communication network [5]. Routing is just an example of network related matter that is affected by topology [6].

This work proposes a 3-dimensional way for modeling the topology. The approach of developing the proposed model makes use of G.T. ideas, tools, and theorems.

2. Modeling Network Topology and Its Properties with Graph Theory

Graph theory is a branch of mathematics concerned mainly with structures. This theory is concerned with patterns of relationships among pairs of abstract elements [2]. Graph theory has many practical applications in various disciplines like engineering, biology, computer science, economics, mathematics, medicine, and social science [7].

Graphs are excellent modeling tools, and graph theory is quite useful when the main interest is in the structural properties of any empirical system as it provides concepts, theorems, and methods appropriates to the analysis of structures [4] [7].

Graph theory represents structures as graphs which in turn will be represented as matrices. These matrices of the way nodes are connected in the graph are called adjacency matrices. Graph theory could be used to model network topologies, keeping in mind that the topology of a network is in reality a structure, as well as many topology properties and measures [1]. Figure (1) shows a simple network with its corresponding graph and adjacency matrix.

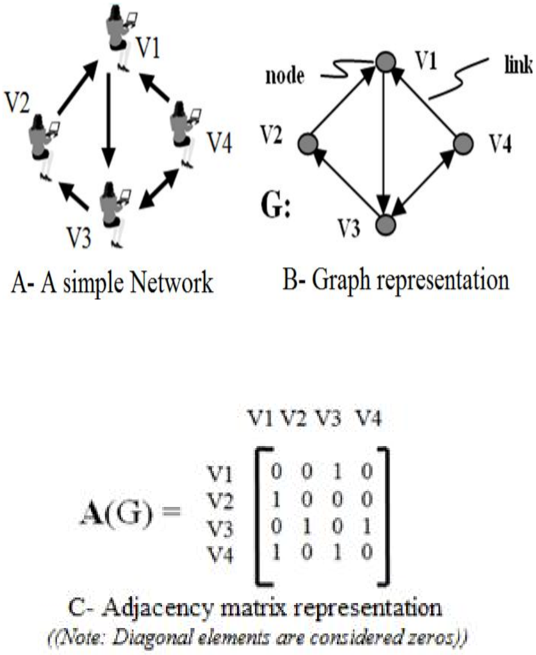
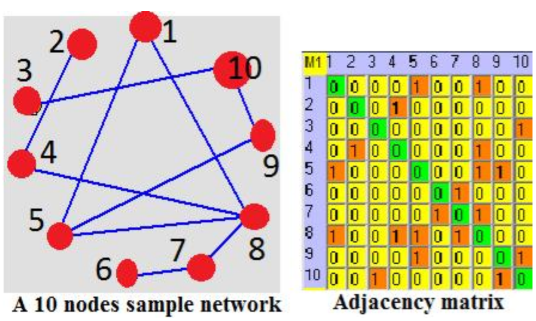


Figure (1): A sample network topology with its representative graph and adjacency matrix.

With graph theory many useful topology properties could be computed. Figure (2) shows a 10 node sample network with its adjacency matrix and two properties, the first of which, the distance (geodesic) matrix, gives the shortest path length between nodes while the other, reachability matrix, shows what a given node can reach of the other nodes[1].



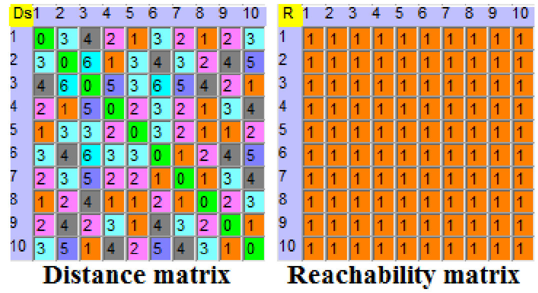


Figure (2): A 10 nodes sample network with its adjacency matrix and two properties.

Another important matrix is the number of geodesic matrix which gives the number of geodesics between each pair of nodes. Figure (3) shows an example for this matrix [1].

All of the matrices given so far are two dimensional arrays. Finding the distance matrix involves in part its repetitive logic multiplication of the adjacency matrix [2]. Each stage of multiplication reveals paths of certain lengths.

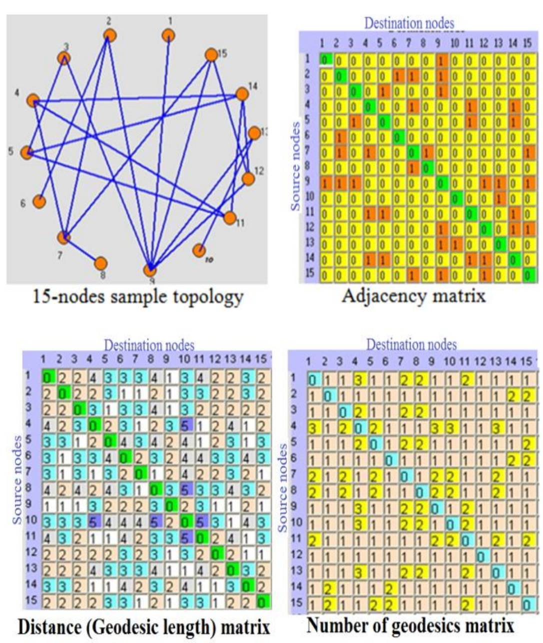
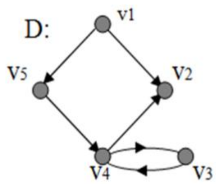


Figure (3): A 15 nodes sample network with its adjacency and geodesics matrices.

For example, logic multiplication of adjacency matrix (A) by itself, i.e. A^2 #, results in a matrix indicating the paths of two links length, and A^3 # indicates the paths of three links length, and so on. But one of the disadvantages with this approach is that it results in invalid paths. For example in Figure (4), A^4 # indicates that there is a path of length four between node 1 and node 4 which moves through nodes this way, 1,5,4,3,4. Of course such a path is not acceptable as part of the path which passes from node 4 to 3 and back to 4 is not needed at all.

Moreover, the matrices produced with the given approach do not tell the nodes through which the path travels through. Also two matrices are needed to indicate the geodesic length and the number of geodesics.



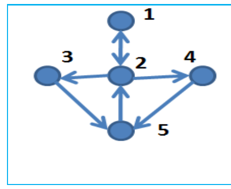
$$A = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{array}{ccccc} V_1 & V_2 & V_3 & V_4 & V_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$A^4\# = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{array}{ccccc} V_1 & V_2 & V_3 & V_4 & V_5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Figure (4): A 5 nodes network with the adjacency matrix and the matrix of Paths of length four (A^4 #).

3. A Proposed 3-Dimensional Modeling Array.

To avoid some limitations of the two dimensional matrix approach mentioned earlier, a three dimensional array is proposed. To explain the difference between the two cases, the sample network of Figure (5) will be used to apply the proposed modeling approach.



	1	2	3	4	5
1	0	1	0	0	0
2	1	0	1	1	0
3	0	0	0	0	1
4	0	0	0	0	1
5	0	1	0	0	0

A sample simple Network — Adjacency matrix A (paths of length 1)

	1	2	3	4	5
1	1		1	1	
2		1			
3		1			
4		1			
5	1		1	1	

	1	2	3	4	5
1		1			1
2	1	1	1	1	
3	1		1	1	
4	1		1	1	
5		1			1

A^2#A^3#

Figure (5): A sample network with the adjacency matrix and its logic powers of 2, and 3.

The proposed method adopts the idea of modeling paths rather than links. But of course the links are still contained within the path models. It also adopts the concept of levels. So A^1 (3,2) (or A(1,3,2)) means paths from node 3 to node 2 of level or length 1, and A^4 (1,5) (or A(4,1,5)) means paths from node 1 to node 5 of length 4, and so on. As it is seen, this is a three dimensional representation of paths. For each level, i.e. path length, each path will be represented by a two dimensional array. Figure (6) gives some examples of path representation regarding the sample network of Figure (5).

	1	2	3	4	5
1		1			
2					
3					
4					
5					

	1	2	3	4	5
1					
2	1				
3					
4					
5					

A^1 (1,2) A^1 (2,1)

	1	2	3	4	5
1					
2					
3					
4					1
5					

	1	2	3	4	5
1					
2					
3					
4					
5					

A^1 (4,5) A^1 (3,2)

Note: empty cells here mean 0 and states for no link existence.

Figure (6) : Representation of some sample paths for the network in Fig.(5).

The proposed approach could be used to represent topology and to compute its properties and goodness measures. Also the model could be used to compute path weights depending on link weights. A sample for such a model for a 5 node network is shown in figure (7). The shown array is of size (4×5×5). For such a network the maximum path length will be 4, and in general the maximum path length equals the number of nodes in the network minus one. Such an array gives the following information:

- Direct links between nodes (i.e. paths of length1). This means that the adjacency matrix will be included in this model.
- Paths existence between nodes and length of such paths and the number of such paths.
- The nodes through which a given path travels.

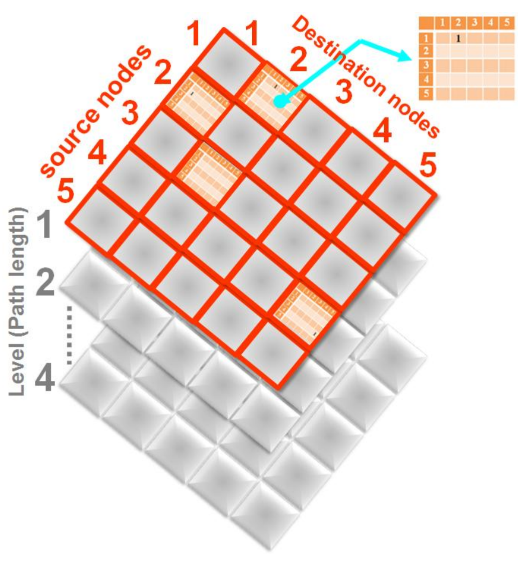


Figure (7): A 3-dimensions array which models topology for a 5 nodes network.

To fill the whole cells of such an array with a network of n nodes, the following algorithm is used:

i. Do a loop for k=2 to n-1
ii. Do a loop for s=1 to n
iii. Do a loop for d=1 to n, s≠d
iv. Find:

$A^k(s,d)=\sum_{i=1}^{n}(A^1(s,i) + A^{k-1}(i,d)) \times (PE A^1 s,i) \times (PE(A^{k-1}(i,d)))$ ……………(1)

Where:

- n= number of nodes in the network.
- S,d: source and destination nodes.
- PE = Path Existence
- PE$A^z(v,w)$means path existence between node v and w with length z and is found using the following formula:$\bigcup_{m=1}^{n}\bigcup_{l=1}^{n} a_{A^z(v,w)}(m,l)$

Where $a_{A^z(v,w)}(m,l)$ is an element of the matrix $A^z(v,w)$defined by row m and column l, and ∪ stands for the OR function.

- Path length from a node to itself is zero.

v. Eliminate possibility that the target path from s to d contains a link from s to d:

$A^k(s,d) = \left(a_{A^k(s,d)}(s,d) NAND\ a_{A^1(s,d)}(s,d)\right) \times A^k(s,d)$ ………. (2)

vi. Eliminate possibility of link repetition in a path:

$A^k(s,d) = \left[\prod_{i=2}^{n}\left(\prod_{j=1}^{i-1} a_{A^k(s,d)}(i,j)\ NAND a_{A^k(s,d)}(j,i)\right)\right] \times \times A^k(s,d)$…….(3)

vii. Back to loop iii.
viii. Back to loop ii.
ix. Back to loop i.
x. End.

In what follows the procedure above will be used to find paths of levels (length) other than 1for some sample cases.

Example ((1)): finding A^2 (5,1).

Step 1: Find preliminary A^2 (5,1):

$A^2(5,1)=\sum_{i=1}^{n}(A^1(5,i) + A^1(i,1)) \times (PEA15,i) \times (PE(A1(i,1)))$….. (4)

Equation 4 then yields:

$A^2(5,1) = 0 + A^1(5,2) + A^1(2,1) + 0 + 0 + 0$

=

	1	2	3	4	5
1					
2					
3					
4					
5		1			

+

	1	2	3	4	5
1					
2	1				
3					
4					
5					

$A^1(5,2)$ $A^1(2,1)$

,and so $A^2(5,1)$ =

	1	2	3	4	5
1					
2	1				
3					
4					
5		1			

The result is very indicative and it indicates that there is single path of length 2 links from node 5 to node 1.

Step 2: Eliminate the possibility that the path contains link 5,1:

$A^2(5,1) = \left(a_{A^2(5,1)}(5,1) NAND\ a_{A^1(5,1)}(5,1)\right) \times A^2(5,1)$ ………. (5)

So: $A^2(5,1) = (0\ NAND\ 0) \times A^2(5,1) = A^2(5,1)$

Step 3: Eliminate possibility of link repetition in the path:

$A^2(5,1) = [\prod_{i=2}^{n}(\prod_{j=1}^{i-1} a_{A^2(5,1)}(i,j)\ NAND a_{A^2(5,1)}(j,i))] \times A^2(5,1)$ …….(6)

And so: $A^2(5,1) = 1\times A^2(5,1) = A^2(5,1)$

Example ((2)): finding $A^2(2,5)$:

Step 1: Find preliminary $A^2(2,5)$:

$A^2(2,5) = \sum_{i=1}^{n}(A^1(2,i) + A^1(i,5)) \times (PEA12,i) \times (PE(A1(i,5)))$

$= 0 + 0 + A^1(2,3) + A^1(3,5) + A^1(2,4) + A^1(4,5) + 0$

=

	1	2	3	4	5
1					
2			1		
3					
4					
5					

+

	1	2	3	4	5
1					
2					
3					1
4					
5					

+

$A^1(2,3)$ $A^1(3,5)$

	1	2	3	4	5
1					
2				1	
3					
4					
5					

+

	1	2	3	4	5
1					
2					
3					
4					1
5					

$A^1(2,4)$ $A^1(4,5)$

=

	1	2	3	4	5
1					
2			1		
3					1
4					
5					

+

	1	2	3	4	5
1					
2				1	
3					
4					1
5					

and so $A^2(2,5)$ =

	1	2	3	4	5
1					
2			1	1	
3					1
4					1
5					

The next steps could be checked as well and still the result will be as it is as there are no eliminations. The result indicates that there are two paths of length 2 links from node 2 to node 5. The first is 2,3,5 and the second is 2,4,5.

Example ((3)): finding $A^3(2,4)$:

Following the same procedure as above we get:

$A^3(2,4) = A^1(2,1) + A^2(1,4)$

=

	1	2	3	4	5
1					
2	1				
3					
4					
5					

+

	1	2	3	4	5
1		1			
2				1	
3					
4					
5					

$A^1(2,1)$ $A^2(1,4)$

=

	1	2	3	4	5
1		1			
2	1			1	
3					
4					
5					

With applying the elimination step $A^3(2,4)$ will be 0 which means that it does not exist as a valid path.

4. Assessments of Results and Conclusions

A three dimensional model has been proposed to model network topology. The model has been developed based on graph theory ideas and theorems. The proposed model adds some important features and provides the following information about topology:

- The adjacency matrix of the network topology defines the direct links and direction between the network nodes.
- The existence of paths between any node pair, and the number of such paths.
- The length of the existence paths.
- The nodes through which a given path could travel starting from a given source node towards a given destination node.

In addition to the features just mentioned, the new model excludes any invalid paths in which a link is traversed more than once.

The introduced model can serve computing topology properties and topology measure as well. The model can be used to compute path weights depending on link weights when they are defined. The formulas developed in the work suits programming purposes, and could be used to develop objective functions for network topology optimization approaches and programs.

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