On Inductively Quasi – \(\hat{g}\) s- Open Functions

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Abstract

The purpose of this paper is to give a new type of open function called inductively quasi – \(\hat{g}\) s-open function, also we obtain its characterization and its basic properties.

1-Introduction

Function and of course open function stand among the most important notions in the whole of mathematical science. Many different forms of open function have been introduced over the year.

In [3] this paper study of related function by involving \(\hat{g}\) s-open sets, and [6] study of quasi \(\hat{g}\) s-open function.

We introduce and characterize the concept of inductively quasi \(\hat{g}\) s-open function.

Throughout this paper, spaces means topological spaces and \(f: (X, T) \rightarrow (Y, V)\) (or simply \(f: X \rightarrow Y\)) denotes a function of space \((X, T)\) into a space \((Y, V)\) also \(f: X \Rightarrow Y\) denotes onto function, and \(\hat{g}\) s-open means (generalization semi-open).

2-Basic Definitions and Theorems

They would like to point out that all the definitions provided in this paper has been formulated by all theirers by adoption of their counterparts in topological spaces.

Definition(2-1)

A subset \(A\) of a space \((X, T)\) is called semi-open (s-open) if \(A \subset \text{cl}(\text{int}(A))\), the complement of s-open set is called s-closed[1] of a subset \(A\) of \(X\),

Denoted by \(s\ cl(A)\), is defined to be the intersection of all s-closed sets containing \(A\) in \(X\).

Definition(2-2)

A subset \(A\) of space \(X\) is called;

i- \(\hat{g}\) -closed [3] if \(\text{cl}(A) \subset U\), whenever \(A \subset U\) and \(U\) is s-open in \(X\). the complement of \(\hat{g}\)-closed set is called \(\hat{g}\) -open set.

ii- \(g^*\) -closed [6] if \(\text{cl}(A) \subset U\), whenever \(A \subset U\) and \(U\) is \(g\)-open in \(X\). the complement of \(g^*\)-closed set is called \(g^*\) -open set.

iii- \(g^*\)-semi-closed[4] if \(\text{scl}(A) \subset U\), whenever \(A \subset U\) and \(U\) is \(g^*\) -open in \(X\). the complement of \(g^*\) -semi-closed set is called \(g^*\) -open set.

iv- \(\hat{g}\) -semi-closed (briefly \(\hat{g}\) s-closed)[10] if \(\text{cl}(A) \subset U\), whenever \(A \subset U\) and \(U\) is \(g^*\) -semi open in \(X\). the complement of \(\hat{g}\) s-closed set is called \(\hat{g}\) s-open set.
The union (respectively intersection) of (openly closed) sets, each contained in (respectively containing) a set $A$ in a spaces $X$ is called the (respecively open - closed) of $A$ and is denoted by $\text{gs-int}(A)$ (resp. $\text{gs-cl}(A)$) [5].

**Definition (2-3)**
A function $f : (X, T) \rightarrow (Y, V)$ is said to be $\text{gs-open}$ [7] if $f(V)$ is $\text{gs-open}$ in $Y$, for every open subset $V$ of $X$.

**Definition (2-4)**
A function $f : (X, T) \rightarrow (Y, V)$ is said to be quasi $\text{gs-open}$ [9] if $f(V)$ is open in $Y$, for every $\text{gs-open}$ subset $V$ of $X$.

**Definition (2-5)**
A function $f : (X, T) \rightarrow (Y, V)$ is said to be inductively open function iff $\exists X_1 \subseteq X$ and $f(X_1) = f(X)$ and $f|_{X_1} : X_1 \rightarrow f(X)$ is open.

**Definition (2-6)**
A function $f : (X, T) \rightarrow (Y, V)$ is said to be inductively quasi $\text{gs-open}$ function iff $\exists X_1 \subseteq X_\varnothing$ and $f(X_1) = f(X)$ and $f|_{X_1} : X_1 \rightarrow f(X)$ is quasi $\text{gs-open}$.

**Definition (2-6) [11]**
Let $f : X \rightarrow Y$ be a function and $A \subseteq X$, a set $A$ is said to be an inverse set iff $A = f^{-1}(f(A))$.

### 3- Basic theorems

**Theorem (3-1)**
If $f : X \rightarrow Y$ is inductively quasi $\text{gs-open}$ function and $A$ be inverse subset of $X$, then $f|_{A} : A \rightarrow Y$ is also inductively quasi $\text{gs-open}$ function.

**Proof**:
Let $f : X \rightarrow Y$ be inductively quasi $\text{gs-open}$ function, then $\exists X_1 \subseteq X\varnothing$ and $f(X_1) = f(X)$ and $f|_{X_1} : X_1 \rightarrow f(X)$ is quasi $\text{gs-open}$.

Now, to show $f|_{A} : A \rightarrow Y$ inductively quasi $\text{gs-open}$ function.

Let $A_1 \subseteq A \varnothing$, $A_1 = A \cap X_1$

we need to show $f(A_1) = f(A) \cap X_1$ and $f|_{A_1} : A_1 \rightarrow f(A)$ is quasi $\text{gs-open}$ function.

$f(A_1) = f(A \cap X_1)$

$= f(f^{-1}(f(A)) \cap X_1)$

$= f(A) \cap f(X_1)$

$= f(A)$

Now, let $W$ be $\text{gs-open}$ in $A_1$

So, $\exists$ $\text{gs-open}$ set $W^* \in X_1 \varnothing$ $W = W^* \cap A_1$

$f(W)$

$= f(W^* \cap A_1)$

$= f(W^* \cap f(A))$

$= f(W^*) \cap f(A)$
Since $W^*$ is $\hat{g}s$-open in $X_1$ and $f\big|_{X_1}: X_1 \to f(X)$ is quasi $\hat{g}s$-open.

Therefore $f(W^*)$ open in $f(X)$

Thus $f\big|_{A_1}: A_1 \to f(A)$ is quasi $\hat{g}s$-open function.

Therefore $f\big|_{A_1}: A_1 \to f(A)$ is quasi $\hat{g}s$-open function.

Theorem(3-2)

If $f: X \Rightarrow Y$ is inductively quasi $\hat{g}s$-open function and $\emptyset \neq T \subseteq Y$, then $f_1: f^{-1}(T) \to T$ be also inductively quasi $\hat{g}s$-open function.

Proof:-

$f: X \Rightarrow Y$ is inductively quasi $\hat{g}s$-open function.

$\exists X_1 \subseteq X \ni f(X_1) = f(X)$ and $f\big|_{X_1}: X_1 \to Y$ is quasi $\hat{g}s$-open.

Now, to prove $f_1: f^{-1}(T) \to T$ inductively quasi $\hat{g}s$-open function.

Let $X^* \subseteq f^{-1}(T) \ni X^* = X_1 \cap f^{-1}(T)$

Now, to show $f_1(X^*) = T$ and $f_1\big|_{X^*}: X^* \to T$ is quasi $\hat{g}s$-open function.

$f_1(X^*) = f(X^*) = f(X_1 \cap f^{-1}(T))$

$= f(X_1) \cap T$

$= Y \cap T$

$= T$

Now, let $W$ be $\hat{g}s$-open in $X^*$

Hence, $\exists$ $g$s-open set $W^*$ in $X_1$ $\ni W^* \cap X^*$

$f(W) = f(W^* \cap X^*) = f(W^*) \cap f^{-1}(T)$

Since $W^*$ is $\hat{g}s$-open in $X_1$ and $f\big|_{X_1}: X_1 \to f(X)$ is quasi $\hat{g}s$-open.

Then $f(W^*)$ open in $Y$.

Therefore $f(W^*) \cap T$ is open in $T$.

Thus $f_1\big|_{X^*}: X^* \to T$ is quasi $\hat{g}s$-open function.

Therefore $f_1: f^{-1}(T) \to T$ inductively quasi $\hat{g}s$-open function.

Theorem(3-3)

Let $f: X \Rightarrow Y$ be a function, $X = W_1 \cup W_2$ with $f(W_1)$ and $f(W_2)$ are open in $f(X)$, if $f\big|_{W_1}: W_1 \to Y$ and $f\big|_{W_2}: W_2 \to Y$ are inductively quasi $\hat{g}s$-open function.

then $f: X \Rightarrow Y$ inductively quasi $\hat{g}s$-open function.

Proof:-

$f\big|_{W_1}: W_1 \to Y$ inductively quasi $\hat{g}s$-open function

then, $\exists X_1 \subseteq W_1 \ni f(X_1) = f(W_1)$ and $f\big|_{X_1}: X_1 \to f(W_1)$ is quasi $\hat{g}s$-open function.

also $f\big|_{W_2}: W_2 \to Y$ is inductively quasi $\hat{g}s$-open function.

$\exists X_2 \subseteq W_2 \ni f(X_2) = f(W_2)$ and $f\big|_{X_2}: X_2 \to f(W_2)$ is quasi $\hat{g}s$-open function.

Now to show $f: X \Rightarrow Y$ is inductively quasi $\hat{g}s$-open function.

Let $X^* = X_1 \cup X_2 \subseteq X$

$$f(X^*) = f(X_1 \cup X_2) = f(X_1) \cup f(X_2)$$
\[ f(W_1) \cup f(W_2) = f(W_1 \cup W_2) = f(X) \]

So \( f(X^*) = f(X) \) and to show \( f \mid x^* \colon X^* \to f(X) \) is quasi \( g_s \) – open function.

Let \( T \) \( g_s \) – open in \( X^* \)

\[ T = T \cap X^* = (T \cap X_1) \cup (T \cap X_2) \]

\[ f(T) = f((T \cap X_1) \cup (T \cap X_2)) \]

\[ f(T) = f(T \cap X_1) \cup f(T \cap X_2) \]

Since \( T \) \( g_s \) – open in \( X^* \), so \( T \cap X_1 \) \( g_s \) – open in \( X_1 \)

And \( f \mid x^* \colon X_1 \to f(W_1) \) is quasi \( g_s \) – open function.

So \( f(T \cap X_1) \) open in \( f(W_1) \), \( f(T \cap X_2) \) open in \( f(X) \).

Similarly \( f(T \cap X_2) \) open in \( f(X) \)

\[ f \mid x^* \colon X^* \to f(X) \) is quasi \( g_s \) – open function.

Therefore \( f : X \to Y \) inductively quasi \( g_s \)-open function.

**Theorem (3-4)**

If \( f : X \to Y \) is inductively quasi \( g_s \) - open function and \( g : Y \to Z \) is open function, then \( g \circ f : X \to Z \) is inductively quasi \( g_s \) - open function.

**Proof**:

since \( f : X \to Y \) is inductively quasi \( g_s \) - open function

then \( \exists X^* \subseteq X \) \( f(X^*) = f(X) \) and \( f \mid x^* \colon X^* \to f(X) \) is quasi \( g_s \) – open function.

now , to prove \( g \circ f : X \to Z \) is inductively quasi \( g_s \) - open function.

we need to show \( g(f(X^*)) = g(f(X)) \) where \( X^* \subseteq X \) and \( g \mid x^* \colon X^* \to Z \) quasi \( g_s \) - open function.

\[ g(f(X^*)) = g[f(X^*)]g[f(X)] = g(f(X)) \]

and let \( U \) be \( g_s \) - open set in \( X^* \), and since \( f \mid x^* \colon X^* \to f(X) \) is quasi \( g_s \) – open function.

then \( f(U) \) is open in \( Y \)

and so and \( g : Y \to Z \) is open function.

then \( g(f(U)) = g(f(U)) \) is open in \( Z \)

so \( g \circ f : X \to Z \) is inductively quasi \( g_s \) - open function.

Therefore \( g \circ f : X \to Z \) is inductively quasi \( g_s \) - open function.

**Theorem (3-5)**

Let \( f : X \to Y \) and \( g : Y \to Z \) be two functions and \( g \circ f : X \to Z \) is inductively quasi \( g_s \) - open function. \( g \) is continuous one to one, then \( f \) is inductively quasi \( g_s \) - open function.

**Proof**:

To prove \( f : X \to Y \) inductively quasi \( g_s \) - open function.

Let \( X_1 \subseteq X \), we need to show \( \exists f(X_1) = f(X) \) and \( f \mid x_1 : X_1 \to f(X) \) is quasi \( g_s \) - open.

Now \( f(X_1) = f(X) \) [since \( g(f(X_1)) = g(f(X)) \)]

let \( G \) be \( g_s \) – open set in \( X_1 \).
gof(G) is open in $Z$[since and g of :$X \to Z$ is inductively quasi $\mathfrak{g}s$ – open function] 

now g :$Y \Rightarrow Z$ is continuous one to one function 
\[ f(G) = g^{-1}(gof(G)) = g^{-1}[g[f(G)]] \] is open in $Y = f(X)$ 
therefore $f : X \to Y$ is inductively quasi $\mathfrak{g}s$ - open function 

References 
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