Theorems on Lorentz Space

Eman Samir Bhaya  
Nada Mohammed Abbas  
*Department Of Mathematics ,College Of Education , Ibin-Hayan Babylon University.*

**Abstract**

In this paper we introduce a generalization of the Lorentz space .And prove some theorems about it.

1-Introduction

Let $f$ be a complex-valued measurable function defined on a finite measure space $(X, \mathcal{A}, \mu)$. For $s \geq 0$, define $\mu^f(s)$ the distribution function of $f$ as

$$
\mu^f(s) = \mu\{x \in X : |f(x)| \geq s\}. \quad [\text{Arora et al.}, \ 2007]
$$

By $f^*$ we mean the non-increasing rearrangement of $f$ given as

$$
f^*(t) = \inf\{s > 0 : \mu^f(s) \leq t\}, \quad t \geq 0.
$$

For $t > 0$, let

$$
\mathcal{F}(t) = \frac{1}{t} \int_0^t f^*(x) \, dx.
$$

For a measurable function $f$ on $X$, define

$$
\|f\|_{p,q} = \left\{ \frac{1}{q} \int_0^\infty \left( \frac{1}{p} \int_0^\infty \left( \frac{1}{2} f^*(s) \frac{ds}{s} \right)^q \frac{ds}{s} \right)^{\frac{1}{q}} \right\}^{\frac{1}{p}} 0 < q, p < 1
$$

The Lorentz space $L(p, q)$ consists of those complex-valued measurable functions $f$ on $X$ such that $\|f\|_{p,q} < \infty$. For more on Lorentz space one can refer to [Bennet and Sharpley1988; Hunt 1966; Lorentz 1950; Stein and Weiss,1971].

Let $T:X \to X$ be a measurable ( $T^{-1}(E) \in \mathcal{A}$, for $E \in \mathcal{A}$) non-singular transformation $\{\mu(T^{-1}(E)) = 0 \text{ whenever } \mu(E) = 0\}$ and $u$ a complex-valued measurable function defined on $X$.

We define a linear transformation $W = W_{u,T}$ on the Lorentz space $L(p, q)$ into the linear space of all complex-valued measurable functions by

$$(W_{u,T}(f))(x) = u(T(x))f(T(x)), x \in X, f \in L(p, q).$$

If $W$ is bounded with range in $L(p, q)$, then it is called a weighted composition operator on $L(p, q)$. if $u \equiv 1$, then $W \equiv C_{f \circ T} f \circ T$ is called composition operator induced by $T$. If $T$ is identity mapping, then $W \equiv M_u : f \rightarrow u \cdot f$, a multiplication operator induced by $u$. The study of these operators on $L_{p,q}$-spaces has been made in [Chan1992; Jabbarzadeh and Pourreza 2003; Jabbarzadeh 2005; Singh and Manhas 1993; Takagi 1993] and references there in .Composition and multiplication operators on the Lorentz spaces were studied in [Kumar,2005, Arora et al., 2006] respectively.

In this paper a characterization of the non-singular measurable transformations $T$ from $X$ into itself and complex-valued measurable function $u$ on $X$ inducing weighted composition operators is obtained on the Lorentz space $L(p, q), 0 < q, p < 1$.

2. Characterizations
In this section we introduce our main results.

**Theorem 2.1.** \( \| \cdot \|_{pq} \) is a quasi-norm on \( L(p, q) \) for \( 0 < q, p < 1 \).

**Proof:**

(i) Assume \( \| f \|_{pq} = 0 \), we must prove \( f = 0 \), to show that it is sufficient to prove,
\[
\int_0^\infty \left( \frac{1}{t^p} \right)^{q} \frac{dt}{t} = 0,
\]
where \( t > 0 \).

So we must show \( f^* (t) = 0 \), i.e.
\[
\frac{1}{t^p} \int_0^t f^*(s) \, ds = 0.
\]

Since \( \frac{1}{t^p} > 0 \), hence it is remain to show \( f^*(s) = 0 \) or \( \inf \{ s > 0 : \mu f(s) \leq t \} = 0 \),
which is clear since \( s > 0 \).

Now, if \( f = 0 \), so \( \mu f(s) = \mu \{ x \in X : 0 > s \} \), which contracts since \( s > 0 \), so there is no \( x \in X \) such that \( |f(x)| = 0 > s \), this leads to \( \mu \{ x \in X : 0 > s \} = 0 \) and
\[
\inf \{ s > 0 : \mu f(s) = 0 \leq t \} = 0
\]
so \( f^*(t) = 0 \), hence \( \| f \|_{pq} = 0 \).

(ii) Since \( |\alpha f(x)| = |\alpha| |f(x)| \), so \( \| \alpha f \|_{pq} = |\alpha| \| f \|_{pq} \).

(iii) To prove the triangular inequality, we must prove
\[
(f + g)^* \leq c(f^* + g^*);
\]
\[
\inf \{ s > 0 : \mu f(s) \leq t \} \leq \inf \{ s > 0 : \mu g(s) \leq t \} + \inf \{ s > 0 : \mu f(s) \leq t \}
\]
and
\[
\mu f + g(s) \leq c(\mu f + \mu g)(s), \quad \ldots (1)
\]
Thus from the definitions of \( \mu f(s), f^*(s), g^*(s) \), it is sufficient to prove (1).

We have
\[
\mu f = \mu \{ x \in X \mid |f(x)| + |g(x)| > s \}
\]
\[
= \mu \{ x \in X : |f(x)| > s_1 \} + \mu \{ x \in X : |g(x)| > s_2 \}
\]
where \( s_1 > s_2 > 0 \), and chosen so that for a given \( \varepsilon > 0 \),
\[
\mu \{ x \in X : |f(x)| > s_1 \} + \frac{\varepsilon}{2} \ldots (2)
\]
And
\[
\mu \{ x \in X : |g(x)| > s_2 \} + \frac{\varepsilon}{2} \ldots (3)
\]
Which are true for any \( \varepsilon > 0 \), so combining between (2) and (3) and (4), to get (1)
where \( c = \max \{ c_1, c_2, c_3 \} 

**Theorem 2.2.** Let \( (X, \mathcal{A}, \mu) \) be a \( \sigma \)-finite measure space and \( u : X \to C \) be a measurable function, let \( T : X \to X \) be a non-singular measurable transformation such that the Radon-Nikodym derivative \( f_T = d(\mu T^{-1})/d\mu \) is in \( L_{\infty}(\mu) \).

Then \( \mathcal{W} f_T : f \to u \circ T \cdot f \circ T \) is bounded on \( L(p, q) \), \( 0 < q, p < 1 \) if \( u \in L_{\infty}(\mu) \).

**Proof:** Suppose \( b = \| f_T \|_m \), then for \( f \in L(p, q) \), the distribution function of \( \mathcal{W} f \) satisfies, where \( \mathcal{W} f = \mathcal{W} u_T : u \to T \cdot f \circ T \), we have
\[
(\mathcal{W} f)^*(t) \leq \| u \|_{\infty} f^*(t/b) \ldots (1) \quad \text{[Arora et al., 2007]}
\]
Then for \( 0 < q, p < 1 \) we have
\[
\| \mathcal{W} f \|_{pq}^q = \frac{q}{p} \int_0^\infty \left( t^p \right)^q \frac{dt}{t}
\]
Then by using (1) we have
\[ \| \mathcal{W}f \|_{L^q_{b^p}} \leq \| u \|_{L^q_{b^p}} \| f \|_{L^\infty} \]

Thus

\[ \| \mathcal{W} \|_{L^q_{b^p}} \leq b^{\frac{1}{p}} \| u \|_{L^\infty} \]

References


