On fuzzy Laplace transforms for fuzzy differential equations of the third order

Hawrra Fadel Muhammad Ali, Amal Khalaf Haydar,
University of Kufa — College Education for Girls

Abstract

In this paper, we introduce a result for fuzzy derivative of the third-order in the sense of H-differentiability, and we find fuzzy Laplace transform for third-order derivative. In addition, we present examples of fuzzy initial value problems of the third-order.

1. Introduction

The concept of fuzzy derivative was first introduced by Chang and Zadeh in 1972 [1] while the term fuzzy differential equation was coined by Kandel and Bratt in 1978 [2]. Puri and Ralescu [3] introduced the concept of the differential of a fuzzy function. Also, many researchers have worked on the theoretical and numerical solutions of fuzzy differential equations such as [4],[5] and [6].

Recently, Allahviranloo and Ahmadi [7] have proposed the fuzzy Laplace transforms for solving first-order fuzzy differential equations. Salahshour and Allahviranloo [8] pointed out that under what conditions the fuzzy-valued functions can possess the fuzzy Laplace transform.

In this paper, we are going to find a result for fuzzy derivative of the third-order and fuzzy Laplace transform for that order are obtained. The structure of this paper is as follows: In Sect.2, some basic concepts are provided. In Sect.3, a result for fuzzy derivative of the third-order and Laplace transform for that order are presented. In Sect.4, we construct a system can be used for solving fuzzy initial value problems (FIVPs) of the third-order with examples. In Sect.5, a conclusion is drawn.

2. Preliminaries

In this section, we give some necessary definitions and concepts which will used in this paper.

Definition 2.1: [4] A fuzzy number \( u \) in parametric form is a pair \((u, \bar{u})\) of functions \( u(r), \bar{u}(r), 0 \leq r \leq 1 \), which satisfy the following requirements:

1. \( u(r) \) is a bounded non-decreasing left continuous function in \((0,1] \), and right continuous at 0.
2. \( \bar{u}(r) \) is a bounded non-increasing left continuous function in \((0,1] \), and right continuous at 0.
3. \( u(r) \leq \bar{u}(r), 0 \leq r \leq 1 \).

A crisp number \( \alpha \) is simply represented by \( u(r) = \bar{u}(r) = \alpha, 0 \leq r \leq 1 \). We recall that for \( a < b < c \) which \( a, b, c \in \mathbb{R} \), the triangular fuzzy number \( u = (a,b,c) \) determined by \( a,b,c \) is given such that \( u(r) = a + (b-c)r \) and \( \bar{u}(r) = c - (c-b)r \) are the endpoints of the \( r \)-level sets, for all \( r \in [0,1] \).
Definition 2.2 Let \( x, y \in E \). If there exists \( z \in E \) such that \( x = y + z \), then \( z \) is called the H-difference of \( x \) and \( y \), and is denoted by \( x \Theta y \). In this paper, the sign \( \Theta \) always stands for H-difference, and it is denoted by \( x \Theta y \neq x + (-y) \).

Definition 2.3: A fuzzy function \( F : U \rightarrow F_0(\mathbb{R}^n) \) is called H-differentiable at \( x_0 \in U \) if there exists \( DF(x_0) = F'(x_0) \in F_0(\mathbb{R}^n) \), such that the limits
\[
\lim_{h \to 0^+} [(F(x_0 + h) \Theta F(x_0))/h] \\
\lim_{h \to 0^+} [(F(x_0) \Theta F(x_0 - h))/h]
\]
both exist and are equal to \( DF(x_0) \).

Theorem 2.4: Let \( f(x) \) be a fuzzy-valued function on \( [a, \infty) \) represented by \( (f(x), \bar{f}(x)) \). For any fixed \( r \in [0,1] \), assume \( f(x, r) \) and \( \bar{f}(x, r) \) are Riemann integrable on \( [a, b] \) for every \( b \geq a \), and assume there are two positive \( \underline{M}(r) \) and \( \bar{M}(r) \) such that \( \int_a^b |f(x, r)| \, dx \leq \underline{M}(r) \) and \( \int_a^b |\bar{f}(x, r)| \, dx \leq \bar{M}(r) \) for every \( b \geq a \). Then \( f(x) \) is improper fuzzy Riemann-integrable on \( [a, \infty) \) and the improper fuzzy Riemann integral is a fuzzy number. Furthermore, we have
\[
\int_a^\infty f(x) \, dx = \left( \int_a^\infty f(x, r) \, dx, \int_a^\infty \bar{f}(x, r) \, dx \right).
\]

Definition 2.5: Let \( f(t) \) be continuous fuzzy-value function. Suppose that \( f(x)e^{-sx} \) is improper fuzzy Riemann-integrable on \( [0, \infty) \), then \( \int_0^\infty f(x) e^{-sx} \, dx \) is called fuzzy Laplace transforms and is denoted as
\[
L(f(x)) = \int_0^\infty f(x) e^{-sx} \, dx, \quad (s > 0 \text{ and integer}).
\]
From theorem 2.4, we have
\[
\int_0^\infty f(x) e^{-sx} \, dx = \left( \int_0^\infty f(x, r) e^{-sx} \, dx, \int_0^\infty \bar{f}(x, r) e^{-sx} \, dx \right),
\]
also by using the definition of classical Laplace transform: \( l(f(x,r)) = \int_0^\infty f(x,r) e^{-sx} \, dx \) and \( l(\bar{f}(x,r)) = \int_0^\infty \bar{f}(x,r) e^{-sx} \, dx \)
then, we follow:
\[
L(f(x)) = (l(f(x,r)), l(\bar{f}(x,r))).
\]

3. Third order FIVPs

In this section, we have the following result for third-order derivative under H-differentiability:

Theorem 3.1 Let \( F(t), F'(t), F''(t) \) are differentiable fuzzy-valued functions. Moreover, we denote \( \alpha \)-cut representation of fuzzy-valued function \( F(t) \) with \( [F(t)]^\alpha = [f_\alpha(t), g_\alpha(t)] \), then:
\[
[F''(t)]^\alpha = [f''_\alpha(t), g''_\alpha(t)]
\]
Proof Since \( F(t) \) and \( F'(t) \) are differentiable then we get
\[
[F'(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)],
\]
since \( F''(t) \) is differentiable then by definition 2.3 we get
\[ [F^*(t+h) \Theta F^*(t)]^\alpha = [f^*_a(t+h), g^*_a(t+h)] \Theta [f^*_a(t), g^*_a(t)] \]
\[ = [f^*_a(t+h) - f^*_a(t), g^*_a(t+h) - g^*_a(t)] \]
and
\[ [F^*(t) \Theta F^*(t-h)]^\alpha = [f^*_a(t), g^*_a(t)] \Theta [f^*_a(t-h), g^*_a(t)] \]
\[ = [f^*_a(t) - f^*_a(t-h), g^*_a(t) - g^*_a(t-h)] \]
and, multiplying by \( \frac{1}{h} \), \( h > 0 \) we get:
\[ \frac{1}{h} [F^*(t+h) \Theta F^*(t)]^\alpha = \frac{f^*_a(t+h) - f^*_a(t)}{h}, \frac{g^*_a(t+h) - g^*_a(t)}{h} \]
and
\[ \frac{1}{h} [F^*(t) \Theta F^*(t-h)]^\alpha = \frac{f^*_a(t) - f^*_a(t-h)}{h}, \frac{g^*_a(t) - g^*_a(t-h)}{h} \]

Finally, using the fact that \( h \rightarrow 0 \) on both sides, the proof is completed.

**Theorem 3.2** Suppose that \( g(t), g'(t) \) and \( g''(t) \) are continuous fuzzy-valued functions on \([0, \infty)\) and of exponential order and \( g'''(t) \) is piecewise continuous fuzzy-valued function on \([0, \infty)\) with \( g(t) = (\underline{g}(t, \alpha), \bar{g}(t, \alpha)) \), then
\[ L(g''''(t)) = s^2 L(g'(t)) \Theta s g(0) \Theta s g'(0) \Theta s g''(0) \]
\[ \text{(3.1)} \]

**Proof** First, we state the notations carefully as follows: \( \underline{g}', \underline{g} \) and \( \bar{g}'' \) are the lower endpoints function’s derivatives, \( \underline{g}'', \underline{g} \) and \( \bar{g}''' \) are the upper endpoints function’s derivatives. By using theorem 3.1 we have
\[ L(g''''(t)) = L(\bar{g}'''', t, \alpha) \]
\[ \text{(3.2)} \]
for arbitrary fixed \( \alpha \in [0,1] \). Now, by using the definition of classical Laplace transform we get
\[ l(\underline{g}''''(t, \alpha)) = s^2 l(g'(t, \alpha)) - s^2 g(0, \alpha) - s g'(0, \alpha) - g''(0, \alpha) \]
\[ l(\bar{g}''''(t, \alpha)) = s^2 l(\bar{g}'(t, \alpha)) - s^2 \bar{g}(0, \alpha) - s \bar{g}'(0, \alpha) - \bar{g}''(0, \alpha) \]
Then, equation (3.2) becomes
\[ L(g''''(t)) = (s^2 l(\underline{g}'(t, \alpha)) - s^2 g(0, \alpha) - s g'(0, \alpha) - g''(0, \alpha), s^2 l(\bar{g}'(t, \alpha)) - s^2 \bar{g}(0, \alpha) - s \bar{g}'(0, \alpha) - \bar{g}''(0, \alpha)) \]
\[ = s^2 L(g'(t)) \Theta s g(0) \Theta s g'(0) \Theta g''(0) \]

### 4. Constructing solutions via fuzzy initial value problems

In this section, we consider the fuzzy initial value problem
\[ y''''(t) = f(t, y(t), y'(t), y''(t)) \]
\[ y(0) = (\underline{y}(0, \alpha), \bar{y}(0, \alpha)) \]
\[ y'(0) = (\underline{y}'(0, \alpha), \bar{y}'(0, \alpha)) \]
\[ y''(0) = (\underline{y}''(0, \alpha), \bar{y}''(0, \alpha)) \]
\[ y'''(0) = (\underline{y}'''(0, \alpha), \bar{y}'''(0, \alpha)), 0 \leq \alpha \leq 1. \]

By using fuzzy Laplace transform method we have:
\[ L(y''''(t)) = L(f(t, y(t), y'(t), y''(t)). \]
\[ \text{(4.1)} \]

By using theorem 3.2, equation (4.2) can be written as follows:
\[ s^2 L(y(t)) \Theta s^2 y(0) \Theta s y'(0) \Theta s y''(0) = L(f(t, y(t), y'(t), y''(t)) \]
Thus
To solve the linear system (4.3), for simplicity we assume that:

\[ l(\gamma(t, \alpha)) = H(s, \alpha) \]
\[ l(\bar{\gamma}(t, \alpha)) = K(s, \alpha) \]

where \( H(s, \alpha) \) and \( K(s, \alpha) \) are solutions of system (4.3). By using inverse Laplace transform, \( \gamma(t, \alpha) \) and \( \bar{\gamma}(t, \alpha) \) are computed as follows

\[ \gamma(t, \alpha) = l^{-1}(H(s, \alpha)) \]
\[ \bar{\gamma}(t, \alpha) = l^{-1}(K(s, \alpha)) \]

**Example 4.1** Consider the following FIVP

\[ y'''(t) = 2y''(t) + 3y'(t) \]
\[ y(0) = (3 + \alpha, 5 - \alpha) \]
\[ y'(0) = (-3 + \alpha, -1 - \alpha) \]
\[ y''(0) = (8 + \alpha, 10 - \alpha) \]

We note that

\[ f(t, y(t), y'(t), y''(t)) = 2y''(t) + 3y'(t) = (2y''(t, \alpha) + 3y'(t, \alpha), 2\bar{y}'(t, \alpha) + 3\bar{\gamma}(t, \alpha)) \]
\[ f(t, y(t), y'(t), y''(t), \alpha) = 2y''(t, \alpha) + 3y'(t, \alpha) \]

and

\[ \bar{f}(t, y(t), y'(t), y''(t), \alpha) = 2\bar{y}'(t, \alpha) + 3\bar{\gamma}(t, \alpha) \]

Thus

\[ l(\gamma(t, \alpha)) = 2l(y''(t, \alpha)) + 3l(y'(t, \alpha)) \]
\[ = 2[s^3l(y(t, \alpha)) - sy'(0, \alpha) - y'(0, \alpha)] + 3[s^2l(y(t, \alpha)) - \gamma(0, \alpha)] \]
\[ = (2s^2 + 3s)l(y(t, \alpha)) - (6 + 2\alpha)s - (3 + 5\alpha) \]

(4.4)

\[ l(\bar{\gamma}(t, \alpha)) = 2l(\bar{y}'(t, \alpha)) + 3l(\bar{\gamma}(t, \alpha)) \]
\[ = 2[s^3l(\bar{y}(t, \alpha)) - sy(0, \alpha) - \gamma(0, \alpha)] + 3[s^2l(\bar{y}(t, \alpha)) - \bar{\gamma}(0, \alpha)] \]
\[ = (2s^2 + 3s)l(\bar{y}(t, \alpha)) + (-10 + 2\alpha)s + (5\alpha - 13) \]

(4.5)

Substituting (4.4) and (4.5) in system (4.3) gives

\[ l(\gamma(t, \alpha))(s^3 - 2s^2 - 3s) = (3 + \alpha)s^2 + (-9 - \alpha)s + (5 - 4\alpha) \]
\[ l(\bar{\gamma}(t, \alpha))(s^3 - 2s^2 - 3s) = (5 - \alpha)s^2 + (-11 + \alpha)s + (4\alpha - 3) \]

(4.6)

The solution of system (4.6) is as follows:

\[ l(\gamma(t, \alpha)) = \frac{(3 + \alpha)s^2 + (-9 - \alpha)s + (5 - 4\alpha)}{s^3 - 2s^2 - 3s} \]
\[ = \frac{(3 + \alpha)s^2 + (-9 - \alpha)s + (5 - 4\alpha)}{s(s - 3)(s + 1)} \]

\[ l(\bar{\gamma}(t, \alpha)) = \frac{(5 - \alpha)s^2 + (-11 + \alpha)s + (-3 + 4\alpha)}{s^3 - 2s^2 - 3s} \]
\[ = \frac{(5 - \alpha)s^2 + (-11 + \alpha)s + (-3 + 4\alpha)}{s(s - 3)(s + 1)} \]

After performing partition of fractions yields
By using inverse Laplace transform, we get

\[
\begin{align*}
\bar{y}(t, \alpha) &= \frac{4\alpha - 5}{3} l^{-1}\left(\frac{1}{s}\right) + \frac{17 - 2\alpha}{4} l^{-1}\left(\frac{1}{s + 1}\right) + \frac{5 + 2\alpha}{12} l^{-1}\left(\frac{1}{s - 3}\right) \\
&= \frac{4\alpha - 5}{3} + \frac{17 - 2\alpha}{4} e^{-t} + \frac{5 + 2\alpha}{12} e^{3t}
\end{align*}
\]

Example 4.2 Consider the following FIVP

\[
y''(t) = -y''(t) - 3y'(t)
\]

\[
y(0) = \left(\frac{3}{4} + \frac{1}{4}r, \frac{5}{4}, \frac{1}{4}r\right)
\]

\[
y'(0) = \left(\frac{3}{2} + \frac{1}{2}r, \frac{5}{2}, \frac{1}{2}r\right)
\]

\[
y''(0) = \left(\frac{15}{4} + \frac{1}{4}r, \frac{17}{4}, \frac{1}{4}r\right)
\]

We note that

\[
\begin{align*}
 f(t, y(t), y'(t), y''(t)) &= -y''(t) - 3y'(t) = (-\bar{y}'(t, r) - 3\bar{y}'(t, r), -\bar{y}''(t, r) - 3\bar{y}'(t, r)) \\
 \bar{f}(t, y(t), y'(t), y''(t), r) &= -\bar{y}'(t, r) - 3\bar{y}'(t, r)
\end{align*}
\]

Thus

\[
\begin{align*}
l(f(t, y(t), y'(t), y''(t), r)) &= -l(\bar{y}'(t, r)) - 3l(\bar{y}'(t, r)) \\
&= -[s^2l(\bar{y}'(t, r) - s\bar{y}'(0, r) - \bar{y}'(0, r))] - 3[sll(\bar{y}'(t, r) - \bar{y}(0, r))]
\end{align*}
\]

(4.7)

\[
\begin{align*}
l(\bar{f}(t, y(t), y'(t), y''(t), r)) &= -l(y''(t, r)) - 3l(y'(t, r)) \\
&= -[s^2l(y(t, r) - sy(0, r) - y'(0, r))] - 3[sll(y'(t, r) - y(0, r))]
\end{align*}
\]

(4.8)

Substituting (4.7) and (4.8) in system (4.3) gives

\[
\begin{align*}
s^3l(\bar{y}'(t, r) + (s^2 + 3s)l(\bar{y}'(t, r))) &= \left(\frac{3}{4} + \frac{1}{4}r\right)s^2 + \left(\frac{11}{4} + \frac{1}{4}r\right)s + (10 - r)
\end{align*}
\]

(4.9)

The solution of system (4.9) is as follows:

\[
l(y(t, r)) = \frac{(r + 3)s^4 + (2r + 6)s^3 + 12s^2 - (r + 71)s - 12r - 96}{4s^5 - 4s^3 - 24s^2 - 36s}
\]

255
After performing partition of fractions yields
\[
l(\bar{y}(t, r)) = \frac{4r + 32}{12s} - \frac{2s}{s^2 + s + 3} \cdot \frac{(r - 1)}{12} \cdot \frac{s - 7}{s^2 - s - 3}
\]
\[
l(\bar{y}(t, r)) = \frac{40 - 4r}{12s} - \frac{2s}{s^2 + s + 3} + \frac{(r - 1)}{12} \cdot \frac{s - 7}{s^2 - s - 3}
\]
By using inverse Laplace transform, we get
\[
y(t, r) = \frac{4r + 32}{12} \cdot L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{2s}{s^2 + s + 3}\right) - \frac{r - 1}{12} \cdot L^{-1}\left(\frac{s - 7}{s^2 - s - 3}\right)
\]
\[
y(t, r) = 8 + \frac{r}{3} - 2e^{-\frac{r}{2}} [\cos \sqrt{11} t - \frac{\sqrt{11}}{2} \sin \frac{\sqrt{11}}{2} t] - \frac{r - 1}{12} e^{\frac{r}{2}} [\cosh \frac{\sqrt{13}}{2} t - \sqrt{13} \sinh \frac{\sqrt{13}}{2} t]
\]
\[
\bar{y}(t, r) = \frac{40 - 4r}{12} \cdot L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{2s}{s^2 + s + 3}\right) + \frac{r - 1}{12} \cdot L^{-1}\left(\frac{s - 7}{s^2 - s - 3}\right)
\]
\[
\bar{y}(t, r) = \frac{10 - r}{3} - 2e^{-\frac{r}{2}} [\cos \sqrt{11} t - \frac{\sqrt{11}}{2} \sin \frac{\sqrt{11}}{2} t] + \frac{r - 1}{12} e^{\frac{r}{2}} [\cosh \frac{\sqrt{13}}{2} t - \sqrt{13} \sinh \frac{\sqrt{13}}{2} t]
\]

5. Conclusion
A result for fuzzy derivative of the third-order in the sense of H- differentiability and fuzzy Laplace transform for third-order derivative are presented, then we used these results for solving fuzzy initial value problems of the third-order.

References