Hosoya Polynomials of Steiner Distance of the Sequential Join of Graphs

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Abstract

The Hosoya polynomials of Steiner \( n \)-distance of the sequential join of graphs \( J_3 \) and \( J_4 \) are obtained and the Hosoya polynomials of Steiner \( 3 \)-distance of the sequential join of \( m \) graphs \( J_m \) are also obtained.

Keywords: Steiner \( n \)-distance, Hosoya polynomial, Sequential Join.

1. Introduction

We follow the terminology of [2,3]. For a connected graph \( G=(V,E) \) of order \( p \), the Steiner distance\([5,6,7]\) of a non-empty subset \( S \subseteq V(G) \), denoted by \( d_G(S) \) or simply \( d(S) \), is defined to be the size of the smallest connected subgraph \( T(S) \) of \( G \) that contains \( S \); \( T(S) \) is a tree called a Steiner tree of \( S \). If \( |S|=2 \), then \( d(S) \) is the distance between the two vertices of \( S \). For \( 2 \leq n \leq p \) and \( |S|=n \), the Steiner distance of \( S \) is called Steiner \( n \)-distance of \( S \) in \( G \). The Steiner \( n \)-diameter of \( G \), denoted by \( \text{diam}_n^*G \) or simply \( \delta_n^*(G) \), is defined by:

\[
\text{diam}_n^*G = \max\{d_G(S) : S \subseteq V(G), |S|=n\}.
\]

Remark 1.1. It is clear that

1. If \( n > m \), then \( \text{diam}_n^*G \geq \text{diam}_m^*G \).
2. If \( S' \subseteq S \), then \( d_G(S') \leq d_G(S) \).

The Steiner \( n \)-distance of a vertex \( v \in V(G) \), denoted by \( W_n^*(v,G) \), is the sum of the Steiner \( n \)-distances of all \( n \)-subsets containing \( v \). The sum of
Steiner \( n \)-distances of all \( n \)-subsets of \( V(G) \) is denoted by \( d_n(G) \) or \( W_n^*(G) \).

It is clear that
\[
W_n^*(G) = n^{-1} \sum_{v \in V(G)} W_n^*(v,G).
\]

...(1.1)

The graph invariant \( W_n^*(G) \) is called Wiener index of the Steiner \( n \)-distance of the graph \( G \).

**Definition 1.2**

Let \( C_n^*(G,k) \) be the number of \( n \)-subsets of distinct vertices of \( G \) with Steiner \( n \)-distance \( k \). The graph polynomial defined by
\[
H_n^*(G;x) = \sum_{k=n-1}^{n} C_n^*(G,k)x^k,
\]

...(1.2)

where \( D_n^n \) is the Steiner \( n \)-diameter of \( G \); is called the Hosoya polynomial of Steiner \( n \)-distance of \( G \).

It is clear that
\[
W_n^*(G) = \sum_{k=n-1}^{n} k C_n^*(G,k)
\]

...(1.3)

For \( 1 \leq n \leq p \), let \( C_n^*(u,G,k) \) be the number of \( n \)-subsets \( S \) of distinct vertices of \( G \) containing \( u \) with Steiner \( n \)-distance \( k \). It is clear that
\[
C_1^*(u,G,0) = 1.
\]

Define
\[
H_n^*(u,G;x) = \sum_{k=n-1}^{n} C_n^*(u,G,k)x^k.
\]

...(1.4)

Obviously, for \( 2 \leq n \leq p \)
\[
H_n^*(G;x) = \frac{1}{n} \sum_{u \in V(G)} H_n^*(u,G;x).
\]

...(1.5)

Ali and Saeed [1] were first who studied this distance-based graph polynomial for Steiner \( n \)-distances, and established Hosoya polynomials of Steiner \( n \)-distance for some special graphs and graphs having some kind of regularity, and for Gutman’s compound graphs \( G_1 \cdot G_2 \) and \( G_1 : G_2 \) in terms of Hosoya polynomials of \( G_1 \) and \( G_2 \).

**Definition 1.3**

Let \( G_1, G_2, \ldots, G_m, \ m \geq 2 \), be vertex disjoint graphs. The sequential join of \( G_1, G_2, \ldots, G_m \) is a graph denoted by
\[
J_m = G_1 + G_2 + \ldots + G_m,
\]

and defined by
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\[ V(J_m) = \bigcup_{i=1}^{m} V(G_i), \]

\[ E(J_m) = \bigcup_{i=1}^{m} E(G_i) \bigcup \{uv|u \in V_i \text{ and } v \in V_{i+1}, \text{ for } i = 1, 2, \ldots, m-1\} \]

in which \( V_i = V(G_i) \), as depicted in the following figure.

![Diagram showing \( J_m \)](image)

**Fig. 1.1 \( J_m \)**

It is clear that

\[ p(J_m) = \sum_{i=1}^{m} p_i \cdot q(J_m) = q_m + \sum_{i=1}^{m-1} (q_i + p_i p_{i+1}), \]

in which \( p_i = p(G_i) \) and \( q_i = q(G_i) \).

One can easily see that for \( m \geq 3, \sum_{i=1}^{m} G_i \) is not commutative, that is for \( m=3 \)

\[ G_1 + G_2 + G_3 \neq G_1 + G_3 + G_2. \]

In [8], Saeed obtained the (ordinary) Hosoya polynomials of \( J_m \), and in [7], Herish obtained the Steiner \( n \)-diameter of the sequential join of \( m \) empty graphs and of \( m \) complete graphs. Also, the Hosoya polynomials of Steiner distance of the sequential join of \( m \) empty graphs and of \( m \) complete graphs were obtained. For \( m \geq 3 \) and \( n \geq 2 \), the Steiner \( n \)-diameter of the sequential join of \( m \) complete graphs is given by [7]

\[ \text{diam}^*_n J_m = \begin{cases} 
  m + n - 3, & \text{if } 2 \leq n \leq p_1 + p_m \\
  m + n - 3 - \alpha, & \text{if } p_1 + p_m + 1 \leq n \leq p,
\end{cases} \]

where \( \alpha \) is the smallest integer such that

\[ p_1 + p_m + 1 \leq n \leq p_1 + p_m + \sum_{i=1}^{\alpha} r_i. \]

It is obvious that Eq. 1.6 holds for the sequential join of \( m \) graphs \( J_m \).
In this paper, a generalization of the results obtained in [7] is given. We obtained the Hosoya polynomials of Steiner $n$-distance of $J_3$ and $J_4$; and the Hosoya polynomials of Steiner 3-distance of $J_m$, $m \geq 4$. We also obtained $H_n^*(J_3;x)$, for $n \geq 2$ and $H_n^*(J_m;x)$, for $m \geq 4$, where each of $G_i$, for $i = 1, 2, ..., m$ is a special graph.

2. Hosoya Polynomials of Steiner $n$-Distance of $J_3$ and $J_4$

In this section, we consider $J_m$, for $m=3$ and $m=4$. Let $S$ be any $n$-subset of vertices of $J_m$. Let $B(G_i)$, for $i = 1, 2, ..., m$, be the number of all $n$-subsets $S$ such that $\langle S \rangle$ is connected in $G_i$. The following proposition determines the Hosoya polynomials of Steiner $n$-distance of $J_3$.

**Proposition 2.1.** For $3 \leq n \leq p(= p_1 + p_2 + p_3)$,

$$H_n^*(J_3;x) = C_1 x^{n-1} + C_2 x^n,$$

where

$$C_1 = \binom{n}{p} - \binom{p_1 + p_2}{n} - \binom{p_2}{n} + B(G_1) + B(G_2) + B(G_3),$$

$$C_2 = \binom{p_2}{n} + \binom{p_1 + p_3}{n} - [B(G_1) + B(G_2) + B(G_3)].$$

and

$B(G_1), B(G_2)$ and $B(G_3)$ are as defined above.

**Proof.** It is clear that

$$diam_n^*J_3 = \begin{cases} n, & \text{if } 3 \leq n \leq p_1 + p_3 \\ n-1, & \text{otherwise} \end{cases}.$$

Therefore,

$$H_n^*(J_3;x) = C_1 x^{n-1} + C_2 x^n$$

in which $C_1$ is the number of all $n$-subsets of $V(J_3)$ with Steiner distance equals $n-1$, and $C_2$ is the number of all $n$-subsets of $V(J_3)$ with Steiner distance equals $n$. Therefore,

$$C_2 = \sum_{i=1}^{3} \left( \binom{p_i}{n} - B(G_i) \right) + \sum_{j=1}^{n-1} \binom{p_1}{j} \binom{p_3}{n-j}$$

$$= \binom{p_2}{n} + \binom{p_1 + p_3}{n} - [B(G_1) + B(G_2) + B(G_3)].$$
Now, since
\[ C_1 + C_2 = \binom{p}{n}, \]
therefore
\[ C_1 = \binom{p}{n} - C_2 = \binom{p}{n} - \binom{p_1 + p_3}{n} - \binom{p_2}{n} + B(G_1) + B(G_2) + B(G_3) \]
This completes the proof. ■

The following corollary computes the \( n \)-Wiener index of \( J_3 \).

**Corollary 2.2.** For \( 3 \leq n \leq p (= p_1 + p_2 + p_3) \),
\[ W_n^*(J_3) = n \binom{p}{n} - C_1, \]
where \( C_1 \) is given in Proposition 2.1. ■

Next, we shall find the Hosoya polynomials of Steiner \( n \)-distance of \( J_4 \).

**Proposition 2.3.** For \( 3 \leq n \leq p (= p_1 + p_2 + p_3 + p_4) \),
\[ H_n^*(J_4; x) = C_1x^{n-1} + C_2x^n + C_3x^{n+1}, \]
where
\[
C_1 = \sum_{i=1}^{n-2} \sum_{j=1}^{n-1-i} \left[ \binom{p_1}{i} \binom{p_2}{j} \binom{p_3}{n-i-j} + \binom{p_2}{i} \binom{p_3}{j} \binom{p_4}{n-i-j} \right] + \sum_{i=1}^{n-3} \sum_{j=1}^{n-2-i} \sum_{k=1}^{n-1-j} \left[ \binom{p_1}{i} \binom{p_2}{j} \binom{p_3}{k} \binom{p_4}{n-i-j-k} \right] + \sum_{i=1}^{4} B(G_i)
\]
\[
C_2 = \binom{p}{n} - \sum_{i=1}^{n-2} \sum_{j=1}^{n-1-i} \left[ \binom{p_1}{i} \binom{p_2}{j} \binom{p_3}{n-i-j} \right] - \sum_{i=1}^{n-3} \sum_{j=1}^{n-2-i} \sum_{k=1}^{n-1-j} \left[ \binom{p_1}{i} \binom{p_2}{j} \binom{p_3}{k} \binom{p_4}{n-i-j-k} \right] - \sum_{i=1}^{4} B(G_i)
\]
\[
+ \binom{p_1 + p_2}{n} + \binom{p_2 + p_3}{n} + \binom{p_3 + p_4}{n} - \binom{p_1}{n} - \binom{p_2}{n} - \binom{p_3}{n} - \binom{p_4}{n}.
\]
and
\[ C_3 = \binom{p_1 + p_4}{n} - \binom{p_1}{n} - \binom{p_4}{n}. \]

**Proof.** It is clear that \( n - 1 \leq \text{diam}_n J_4 \leq n + 1 \), therefore the Hosoya polynomials of Steiner \( n \)-distance of \( J_4 \) has the following form
\[ H^*_n(J_4; x) = C_1 x^{n-1} + C_2 x^n + C_3 x^{n+1}. \]

To find \( C_1, C_2 \) and \( C_3 \), let \( S \) be any \( n \)-subset of vertices of \( J_4 \), then we have the following possibilities for the subset \( S \).

(I) \( d(S) = n - 1 \) if and only if \( S \) has any of the following subcases:

1. \( S \) is a subset of \( V_j \), for \( i = 1, 2, 3, 4 \) and \( \langle S \rangle \) is a connected subgraph of \( G_i \). The number of these \( n \)-subsets is given by
   \[ B(G_1) + B(G_2) + B(G_3) + B(G_4). \]

2. \( S \subseteq V_k \cup V_{k+1} \) and \( (S \mid V_k \neq \emptyset \land S \mid V_{k+1} \neq \emptyset) \), \( k = 1, 2, 3 \).
   The number of these subsets \( S \) is given by
   \[ \sum_{i=1}^{n-1} \binom{p_1}{i} \binom{p_2}{n-i} + \binom{p_3}{i} \binom{p_4}{n-i} = \binom{p_1 + p_2}{n} + \binom{p_3 + p_4}{n} - \binom{p_1}{n} - 2 \binom{p_2}{n} - 2 \binom{p_3}{n} - \binom{p_4}{n}. \]

3. \( S \subseteq \bigcup_{i=1}^{3} V_i \land S \mid V_i \neq \emptyset \) or \( S \subseteq \bigcup_{i=2}^{4} V_i \land S \mid V_i \neq \emptyset \). The number of these \( n \)-subsets is given by
   \[ \sum_{i=1}^{n-2} \sum_{j=1}^{n-1} \left[ \binom{p_1}{i} \binom{p_2}{j} \binom{p_3}{n-i-j} + \binom{p_1}{j} \binom{p_2}{i} \binom{p_4}{n-i-j} \right]. \]

4. \( S \mid V_i \neq \emptyset \), \( i = 1, 2, 3, 4 \). The number of these \( n \)-subsets is given by
   \[ \sum_{i=1}^{n-3} \sum_{j=1}^{n-2} \sum_{k=1}^{n-1} \binom{p_1}{i} \binom{p_2}{j} \binom{p_3}{k} \binom{p_4}{n-i-j-k}. \]

From (I), (2), (3) and (4), we get \( C_1 \) as given in the statement of the proposition.

(II) \( d(S) = n + 1 \) if and only if \( S \subseteq V_1 \cup V_4 \) and \( (S \mid V_1 \neq \emptyset \land S \mid V_4 \neq \emptyset) \). The number of these \( S \)’s is given by
\[ \sum_{i=1}^{n-1} \binom{p_1 + p_4}{n-i} = \binom{p_1 + p_4}{n} - \binom{p_1}{n} - \binom{p_4}{n}. \]

So, \( C_3 \) is as given.
Now, since \( C_1 + C_2 + C_3 = \binom{p}{n} \),

therefore

\[
C_2 = \binom{p}{n} - C_1 - C_3.
\]

This completes the proof. ■

**Remark.** The triple summation in \( C_1 \) is taken to be zero when \( n=3 \).

The following corollary computes \( W_n^*(J_4) \).

**Corollary 2.4.** For \( 3 \leq n \leq p= (p_1 + p_2 + p_3 + p_4) \),

\[
W_n^*(J_4) = n \left( \binom{p}{n} - C_1 + C_3 \right),
\]

where \( C_1 \) and \( C_3 \) are given in Proposition 2.3. ■

**Remark.** For \( m \geq 5 \), the calculation of the coefficients of \( H_n^*(J_m; x) \) is complicated.

The numbers \( B(G_1), B(G_2) \) and \( B(G_3) \) are given in Proposition 2.1 can be counted for some specific graphs \( G_1, G_2 \) and \( G_3 \) as in the following examples.

**Example 2.5.** Let \( N_{p_1}, N_{p_2} \) and \( N_{p_3} \) be empty graphs of orders \( p_1, p_2 \) and \( p_3 \) respectively, then \( B(N_{p_i}) = B(N_{p_2}) = B(N_{p_3}) = 0 \).

**Example 2.6.** Let \( K_{p_1}, K_{p_2} \) and \( K_{p_3} \) be complete graphs of orders \( p_1, p_2 \) and \( p_3 \) respectively, then

\[
B(K_{p_i}) = \binom{p_i}{n}, \text{ for } i = 1, 2, 3.
\]

**Example 2.7.** Let \( P_{\alpha_1}, P_{\alpha_2} \) and \( P_{\alpha_3} \) be path graphs of orders \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) respectively, then

\[
B(P_{\alpha_i}) = \alpha_i - n + 1, \text{ for } i = 1, 2, 3.
\]

**Example 2.8.** Let \( C_{\alpha_1}, C_{\alpha_2} \) and \( C_{\alpha_3} \) be cycle graphs of orders \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) respectively, then

\[
B(C_{\alpha_i}) = \alpha_i, \text{ for } i = 1, 2, 3.
\]
Example 2.9. Let \( W_{\alpha_1}, W_{\alpha_2}, \) and \( W_{\alpha_3} \) be wheel graphs of orders \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) respectively, then
\[
B(W_{\alpha_i}) = \frac{\alpha_i - 1}{n - 1} + \alpha_i - 1, \text{ for } i = 1, 2, 3.
\]

Example 2.10. Let \( K_{\alpha_i, \beta_i} \), for \( i = 1, 2, 3 \), be complete bipartite graphs of partite sets of size \( \alpha_i, \beta_i \), then
\[
B(K_{\alpha_i, \beta_i}) = \binom{\alpha_i + \beta_i - 1}{n} - \binom{\alpha_i - 1}{n} - \binom{\beta_i - 1}{n}, \text{ for } i = 1, 2, 3.
\]

3. Hosoya Polynomials of Steiner 3-Distance of \( J_m \) (\( m \geq 5 \))

In this section, we consider \( J_m = G_1 + G_2 + \ldots + G_m \), for \( m \geq 5 \). The following theorem determines Hosoya polynomials of Steiner 3-distance of \( J_m \).

Theorem 3.1. For \( m \geq 5 \),
\[
H^*_J(J_m; x) = (A + Bx)x^2 + \frac{1}{2} \sum_{j=1}^{m-1} \sum_{i=1}^{j-1} \sum_{i=1}^{j-1} p_i p_j (p_i + p_j - 2)x^{j-i+1} + \sum_{j=i+2}^{m} \sum_{i=1}^{j-2} \sum_{i=1}^{j-2} p_i p_j \sum_{r=1}^{j-2} x^{j-i},
\]
where
\[
A = \sum_{i=1}^{m} \left[ \sum_{v \in V_i} \binom{\deg v - 2}{2} - 2\gamma_i \right], \quad B = \sum_{i=1}^{m} \binom{p_i}{3} - A,
\]
in which \( \gamma_i \), for \( i = 1, 2, \ldots, m \) is the number of non-identical triangles \( K_3 \) as a subgraph in \( G_i \).

**Proof.** Let \( S \) be any 3-subset of vertices of \( J_m \), then we have three main cases for the subset \( S \).

(I) If \( S \subseteq V_i \), for \( i = 1, 2, \ldots, m \), then
(a) \( d(S) = 2 \), when \( \langle S \rangle \) is a connected subgraph in \( G_i \), and by Lemma 3.4.4. of [7], the number of such 3-subsets \( S \) is given by
\[
A = \sum_{i=1}^{m} \left[ \sum_{v \in V_i} \binom{\deg v - 2}{2} - 2\gamma_i \right].
\]
(b) \( d(S) = 3 \), when \( \langle S \rangle \) is a disconnected subgraph in \( G_i \), and the number of such 3-subsets \( S \) is given by
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\[ B = \sum_{i=1}^{m} \binom{p_i}{3} - A. \]

Case (I) produces the polynomial
\[ F_1(x) = (A + Bx)x^2. \]

(II) Either two vertices of S are in \( V_i \) and one vertex of S in \( V_j \), \( i < j \), or one vertex of S in \( V_i \), and two vertices of S in \( V_j \), for \( 1 \leq i < j \leq m \). For each such case of S,
\[ d(S) = j - i + 1, \]
and the number of ways of choosing such S is given by
\[ \sum_{j=i+1}^{m} \sum_{i=1}^{m-1} \left[ \binom{p_i}{2} p_j + \binom{p_j}{2} p_i \right], \]
and, this produces the polynomial
\[ F_2(x) = \frac{1}{2} \sum_{j=i+1}^{m} \sum_{i=1}^{m-1} (p_j p_i (p_i - 1) + p_i p_j (p_j - 1)) x^{j-i}, \]
\[ = \frac{1}{2} \sum_{j=i+1}^{m} \sum_{i=1}^{m-1} p_i p_j (p_i + p_j - 2) x^{j-i+1}. \]

(III) One vertex of S in \( V_i \), one vertex in \( V_j \), \( j \geq i + 2 \), and the third vertex in \( V_r \), \( i < r < j \). For such case
\[ d(S) = j - i, \]
and the number of all possibilities of such S is
\[ \sum_{j=i+2}^{m} \sum_{i=1}^{m-2} p_i p_j \left( \sum_{r=i+1}^{j-1} p_r \right), \]
and this produces the polynomial
\[ F_3(x) = \sum_{j=i+2}^{m} \sum_{i=1}^{m-2} p_i p_j \left( \sum_{r=i+1}^{j-1} p_r \right) x^{j-i}. \]

Now adding the polynomials \( F_1(x), F_2(x) \) and \( F_3(x) \) obtained in (I), (II) and (III), we get the required result. \( \blacksquare \)

The numbers \( A \) and \( B \) are given in Theorem 3.1 can be counted when \( G_i \), for \( i = 1, 2, \ldots, m \), has a special form, as in the following examples.

Example 3.2. Let \( N_{p_i} \), for \( i = 1, 2, \ldots, m \) be empty graphs of orders \( p_i \), then
\[ A = 0 \quad \text{and} \quad B = \sum_{i=1}^{m} \binom{p_i}{3}. \]
Example 3.3. Let $K_{p_i}$ for $i = 1, 2, \ldots, m$ be complete graphs of orders $p_i$, then

$$A = \sum_{i=1}^{m} \binom{p_i}{3} \quad \text{and} \quad B = 0.$$ 

Example 3.4. Let $P_{\alpha_i}$ for $i = 1, 2, \ldots, m$ be path graphs of orders $\alpha_i$, then

$$A = \sum_{i=1}^{m} [\alpha_i - 2] = p - 2m \quad \text{and} \quad B = \sum_{i=1}^{m} \binom{\alpha_i}{3} - p + 2m.$$ 

Example 3.5. Let $C_{\alpha_i}$ for $i = 1, 2, \ldots, m$ be cycle graphs of orders $\alpha_i$, then

$$A = \sum_{i=1}^{m} \alpha_i = p \quad \text{and} \quad B = \sum_{i=1}^{m} \binom{\alpha_i}{3} - p.$$ 

Example 3.6. Let $W_{\alpha_i}$ for $i = 1, 2, \ldots, m$ be wheel graphs of orders $\alpha_i$, then

$$A = \sum_{i=1}^{m} \left[ \sum_{r \geq 1} \left( \binom{\deg r}{2} - 2 \gamma_i \right) \right] = \sum_{i=1}^{m} \left[ (\alpha_i - 1) \binom{3}{2} + \binom{\alpha_i}{2} - 2(\alpha_i - 1) \right]$$

$$= \sum_{i=1}^{m} \binom{\alpha_i}{2},$$

and

$$B = \sum_{i=1}^{m} \binom{\alpha_i}{3} - \sum_{i=1}^{m} \binom{\alpha_i}{2} = \frac{1}{6} \sum_{i=1}^{m} \alpha_i (\alpha_i - 1)(\alpha_i - 5).$$ 

Example 3.7. Let $K_{\alpha_i, \beta_i}$ for $i = 1, 2, \ldots, m$, be complete bipartite graphs of partite sets of size $\alpha_i, \beta_i$, then it is known that $K_{\alpha_i, \beta_i}$ contains no odd cycles, and so $\gamma_i = 0$, for $i = 1, 2, \ldots, m$.

Hence,

$$A = \sum_{i=1}^{m} \left[ \alpha_i \binom{\beta_i}{2} + \beta_i \binom{\alpha_i}{2} \right] = \frac{1}{2} \sum_{i=1}^{m} \alpha_i \beta_i (\alpha_i + \beta_i - 2),$$

and

$$B = \sum_{i=1}^{m} \left\{ \binom{\alpha_i + \beta_i}{3} - \frac{1}{2} \alpha_i \beta_i (\alpha_i + \beta_i - 2) \right\}.$$
REFERENCES


