# The n-Wiener Polynomials of the Cartesian Product of a Complete Graph with some Special Graphs

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**الملخص**

تضمن هذا البحث إيجاد متعددات وينر-n لجداءات بيان تام $K_t$ مع بيانات خاصة مثل البيان التام $K_r$ ، نجمة $S_r$ ، وبيان ثنائي التجزئة تام $K_{s,r}$ ، وعجلة $W_r$ ودرب $P_r$ . كما تضمن إيجاد قطر-n ودليل وينر-n لكل من الجداءات $K_t \times K_r$ ,$K_t \times S_r$ ,$K_t \times K_{s,r}$ ,$K_t \times W_r$ ,$K_t \times P_r$.

## ABSTRACT

The n-Wiener polynomials of the Cartesian products of a complete graph $K_t$ with another complete graph $K_r$, a star graph $S_r$, a complete bipartite graph $K_{r,s}$, a wheel $W_r$, and a path graph $P_r$ are obtained in this paper. The n-diameters and the n-Wiener indices of $K_t \times K_r$, $K_t \times S_r$, $K_t \times K_{r,s}$, $K_t \times W_r$ and $K_t \times P_r$ are also obtained.

Keywords: n-distance, n-diameter, n-index, n-Wiener polynomial.

## 1. Introduction.

We follow the terminology of [5] and [6]. Let v be a vertex of a connected graph G and let S be an (n-1)-subset of vertices of V(G), $n \geq 2$, then the **n-distance** $d_n(v,S)$ is defined as follows[7]

$$d_n(v,S)=\min\{d(v,u):u\in S\}. \qquad ...(1.1)$$

Sometimes, we refer to the n-distance of the pair (v,S) in G by $d_n(v,S \mid G)$. The **n-diameter** $diam_n G$ of G is defined by

$$diam_n G=\max\{d_n(v,S): v\in V(G), S\subseteq V(G), |S|=n-1\}. \qquad ...(1.2)$$

It is clear that for all $2 \leq m \leq n \leq p$,

$$diam_n G \leq diam_m G \leq diam G. \qquad ...(1.3)$$

The **n-Wiener index** of G denoted by $W_n(G)$ is defined as

$$W_n(G)=\sum_{(v,S)} d_n(v,S), \qquad ...(1.4)$$

where the summation is taken over all pairs (v,S) for which $v\in V(G)$, $S\subseteq V(G)$ and $|S|=n-1$. The **n-average distance** $\mu_n(G)$ is defined as

$$\mu_n(G)= W_n(G)/p\binom{p-1}{n-1},\ \ 3\le n\le p. \qquad \text{…(1.5)}$$

Let v be any vertex of G, then **the n-distance of v** denoted $d_n(v|G)$ or simply $d_n(v)$ is defined as

$$d_n(v)=\sum_{S\subseteq V(G)} d_n(v,S)\ \ , |S| =n-1. \qquad \text{…(1.6)}$$

The Wiener polynomial of G with respect to the n-distance, which is called n-Wiener polynomial and defined as below.

**Definition 1.1.[2].** Let $C_n(G,k)$ be the number of pairs (v,S), $|S|=n-1, 3\le n\le p$, such that $d_n(v,S)=k$, for each $0\le k\le \delta_n$**.** Then, **the n-Wiener polynomial** $W_n(G;x)$ is defined by

$$W_n(G;x)=\sum_{k=0}^{d_n} C_n(G,k)x^k , \qquad \text{…(1.7)}$$

in which $\delta_n$ is the n-diameter of G .

One may easily see [2] that for $3\le n\le p$, the number of all (v,S) pairs is $p\binom{p}{n-1}$, and [1]

$$\sum_{k=1}^{d_n} C_n(G,k)=p\binom{p-1}{n-1}, \qquad C_n(G,0)=p\binom{p-1}{n-2}, \qquad \text{…(1.8)}$$

$$C_n(G,1)=p\binom{p-1}{n-1}-\sum_{v\in V(G)}\binom{p-1-\deg_G(v)}{n-1} \qquad \text{…(1.9)}$$

**Definition 1.2[1]** Let v be a vertex of G, and let $C_n(v,G,k)$ be the number of (n-1)-subsets of vertices of G such that

$d_n(v,S|G)=k$ , for $n\ge 3$, $0\le k\le \delta_n$.

Then, the **n-Wiener polynomial of vertex v**, denoted by $W_n(v,G;x)$ is defined as

$$W_n(v,G;x)=\sum_{k\ge 0} C_n(v,G,k)x^k. \qquad \text{…(1.10)}$$

It is clear that for all $k\ge 0$,

$$\sum_{v\in V(G)} C_n(v,G,k)=C_n(G,k), \qquad \text{…(1.11)}$$

and

$$\sum_{v\in V(G)} W_n(v,G,x)=W_n(G;x)\ . \qquad \text{…(1.12)}$$

There are many classes of graphs G in which for each k, $1\le k\le \delta_n$ , $C_n(v,G,k)$ is the same for every vertex $v\in V(G)$; such graphs are called [1] **vertex-n-distance regular**. If G is of order p and it is vertex-n-distance regular, then

$$W_n(G;x)=pW_n(v,G;x), \quad \ldots(1.13)$$

where v is any vertex of G.

The authors of references [2],[3] and [4] obtained the n-Wiener polynomials of some special graphs and some types of composite graphs. In this paper, we obtain n-Wiener polynomials of the Cartesian products $K_t\times K_r$, $K_t\times S_r$, $K_t\times K_{r,s}$, $K_t\times W_r$ and $K_t\times P_r$.

## 2. The Cartesian Product of a Complete Graph and a Star

Let $K_t$ be a complete graph with $V(K_t)=\{u_1,u_2,\ldots,u_t\}$, and $S_r$ be a star of center $v_0$ and end vertices $v_1,v_2,\ldots,v_{r-1}$. Each vertex of $K_t\times S_r$ is an ordered pair $(u_i,v_j)$, $1\le i\le t$, $0\le j\le r-1$. Let $K_t^j$ be the clique graph [6] of order t of vertex set $\{(u_i,v_j): i=1,2,\ldots,t, 0\le j\le r-1\}$. The graph $K_t\times S_r$ is depicted in Fig. 2.1.

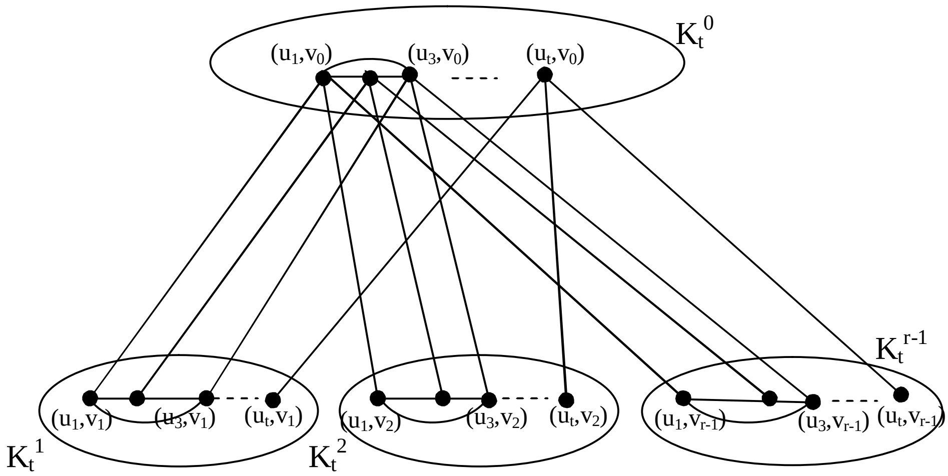


Fig. 2.1. The graph $K_t\times S_r$.

It is clear that $0\le d((u_i,v_j),(u_l,v_m))\le 3$. Thus,

$$\operatorname{diam}_n K_t\times S_r\le \operatorname{diam} K_t\times S_r \le 3.$$

**Proposition 2.1**. For $t\ge 2$, $r\ge 3$, the n-diameter of $K_t\times S_r$ is given by

$$\operatorname{diam}_n K_t\times S_r=\begin{cases} 3, & \text{if } 2\le n\le (t-1)(r-2)+1, \\ 2, & \text{if } 1+(t-1)(r-2)<n\le t(r-1), \\ 1, & \text{if } t(r-1)<n\le rt. \end{cases}$$

**Proof.** The proof is clear from Fig. 2.1. ■

The following theorem gives us the n-Wiener polynomial of $K_t\times S_r$. It is clear that the order of $K_t\times S_r$ is p=rt.

**Theorem 2.2.** For $t\geq 2$, $r\geq 3$, $3\leq n\leq rt$,

$$W_n(K_t\times S_r;x)=p\binom{p-1}{n-2}+[p\binom{p-1}{n-1}-t(r-1)\binom{p-t-1}{n-1}-t\binom{p-t-r+1}{n-1}]x$$
$$+t[\binom{p-t-r+1}{n-1}+(r-1)\{\binom{p-t-1}{n-1}-\binom{p-2t-r+2}{n-1}\}]x^2$$
$$+t(r-1)\binom{p-2t-r+2}{n-1}x^3.$$

**Proof.** It is clear that each vertex of $K_t^0$ is of degree r+t-2, and each vertex of $K_t^j$, $1\leq j\leq r-1$, is of degree t. Therefore, by (1.9) we obtained $C_n(K_t\times S_r,1)$ as given in the theorem.
To find $C_n(K_t\times S_r,3)$, we notice that there are (t-1)(r-2) vertices each of distance 3 from each vertex (u,v) of $K_t^j$, $1\leq j\leq r-1$. Thus,

$$C_n(K_t\times S_r,3)=t(r-1)\binom{(t-1)(r-2)}{n-1}.$$

Finally, by (1.8) and Proposition 2.1, we get

$$C_n(K_t\times S_r,2)=p\binom{p-1}{n-1}-C_n(K_t\times S_r,1)-C_n(K_t\times S_r,3)$$
$$=t[\binom{p-t-r+1}{n-1}+(r-1)\binom{p-t-1}{n-1}-(r-1)\binom{p-2t-r+2}{n-1}].$$

Hence, the proof.■

**Corollary 2.3.** For $t\geq 2$, $r\geq 3$, $3\leq n\leq rt$,

$$W_n(K_t\times S_r)=p\binom{p-1}{n-1}+t[(r-1)\binom{p-t-1}{n-1}+\binom{p-t-r+1}{n-1}+(r-1)\binom{p-2t-r+2}{n-1}].\ ■$$

## 3. The Cartesian Product of Complete Graphs

Let $K_t$ and $K_r$ be disjoint complete graphs, and let $(u_1,v_1),(u_2,v_2)\in V(K_t\times K_r)$, then it is clear that

$$\text{diam } K_t\times K_r=2.$$

Thus,

$$\text{diam}_n\ K_t\times K_r\leq 2,\quad 2\leq n\leq rt.$$

If $u_1\neq u_2$ and $v_1\neq v_2$, then $(u_1,v_1)$, $(u_2,v_2)$ are non-adjacent in $K_t\times K_r$; and $(u_1, v_1)$, $(u_1,v_2)$, $(u_2,v_2)$ is a path of length 2. Therefore,

$$d((u_1,v_1),(u_2,v_2))=2.$$

The degree of each vertex $(u_1,v_1)$ is r+t-2. Thus, the number of vertices of distance 2 from $(u_1,v_1)$ is rt-r-t+1. Hence, we have the following result.

**Proposition 3.1.** For $t,r\geq 2$,

$$\operatorname{diam}_n K_t\times K_r=\begin{cases} 2 & \text{if } 2\leq n\leq rt-r-t+2, \\ 1 & \text{if } rt-r-t+3\leq n\leq rt. \end{cases}$$

Now, we find the n-Wiener polynomial of $K_t\times K_r$.

**Theorem 3.2.** For $r,t\geq 2$, $3\leq n\leq rt$

$$W_n(K_t\times K_r;x)=rt\binom{rt-1}{n-2}+rt\left[\binom{rt-1}{n-1}-\binom{rt-r-t+1}{n-1}\right]x+rt\binom{rt-r-t+1}{n-1}x^2.$$

**Proof.** It is clear that $K_t\times K_r$ is vertex-n-distance regular. Thus,

$$C_n(K_t\times K_r,2)=rtC_n((u_1,v_1), K_t\times K_r,2).$$

Since the number of vertices of distance 2 from $(u_1,v_1)$ is rt-r-t+1, and there is no vertex of distance more than 2 from $(u_1,v_1)$, then

$$C_n((u_1,v_1), K_t\times K_r,2)=\binom{rt-r-t+1}{n-1}.$$

The constant term and the coefficient of x follow from (1.8) and (1.9).■

**Corollary 3.3.** For $r,t\geq 2$, $3\leq n\leq rt$,

$$W_n(K_t\times K_r)=rt\left[\binom{rt-1}{n-1}+\binom{rt-r-t+1}{n-1}\right].■$$

## 4. The Cartesian Product of a Complete Graph and a Complete Bipartite Graphs

Let $K_{r,s}$ be a complete bipartite graph of bipartite sets of vertices $V_1=\{v_1,v_2,\ldots,v_r\}$, $V_2=\{w_1,w_2,\ldots,w_s\}$; $r\geq s$, and let

$$V(K_t)=\{u_1,u_2,\ldots,u_t\},$$

then it is clear that in $K_t\times K_{r,s}$

$$d((u_i,v_h),(u_j,v_k))=3 \text{ when } i\neq j,\ h\neq k,$$

because there is a shortest path

$$(u_i,v_h),\ (u_j,v_h),\ (u_j,w),\ (u_j,v_k),\ w\in V_2.$$

Similarly,

$$d((u_i,w_h),(u_j,w_k))=3 \text{ when } i\neq j,\ h\neq k.$$

Moreover,

$$d((u_i,v_h),(u_i,v_k))= d((u_i,w_h),(u_i,w_k))=2.$$

Therefore,

$$\operatorname{diam} K_t\times K_{r,s}=3,$$

and so

$$\operatorname{diam}_n K_t\times K_{r,s}\leq 3,\ 2\leq n\leq p,\ p=t(r+s).$$

For any vertex $(u_i,v_h)$, the number of vertices of distance 3 from $(u_i,v_h)$ in $K_t\times K_{r,s}$ is (t-1)(r-1). Similarly, there are (t-1)(s-1) vertices of distance 3 from $(u_i,w_k)$. Moreover, the degree of each vertex of $K_t\times K_{r,s}$ is either r+t-1 or s+t-1.

Thus, we have the following result.

**Proposition 4.1.** For t,r,s≥2, r≥s, then the n-diameter of $K_t\times K_{r,s}$ is given

$$\operatorname{diam}_n K_t\times K_{r,s}=\begin{cases}3, & \text{for } 2\leq n\leq tr-t-r+2,\\ 2, & \text{for } tr-t-r+3\leq n\leq p-t-s,\\ 1, & \text{for } p-t-s+1\leq n\leq p.\end{cases}$$

■

The next theorem determines the n-Wiener polynomial of $K_t\times K_{r,s}$.

**Theorem 4.2.** For t,r,s≥2, 3≤n≤p, p=t(r+s),

$$W_n(K_t\times K_{r,s};x)=p\binom{p-1}{n-2}+[p\binom{p-1}{n-1}-st\binom{p-r-t}{n-1}-rt\binom{p-s-t}{n-1}]x$$
$$+\{rt[\binom{p-s-t}{n-1}-\binom{rt-t-r+1}{n-1}]+st[\binom{p-r-t}{n-1}-\binom{ts-t-s+1}{n-1}]\}x^2$$
$$+[rt\binom{rt-t-r+1}{n-1}+st\binom{ts-t-s+1}{n-1}]x^3.$$

**Proof.** $C_n(K_t\times K_{r,s},0)$ and $C_n(K_t\times K_{r,s},1)$ are obtained from (1.8) and (1.9). To find the other coefficients, we notice that $C_n((a,b), K_t\times K_{r,s},k)$ is the same for every vertex $(a,b)\in V(K_t)\times V_1$, and $C_n((c,d),K_t\times K_{r,s},k)$ is the same for every vertex $(c,d)\in V(K_t)\times V_2$, for k=2,3. Since the number of vertices of distance 3 from vertex (a,b) is (t-1)(r-1), and the number of vertices of distance 3 from vertex (c,d) is (t-1)(s-1), then we get the coefficient of $x^3$ as given in the statement of the theorem.

Finally, $C_n(K_t\times K_{r,s},2)$ is obtained using the relation (1.8) and the coefficients already obtained. This completes the proof. ■

**Corollary 4.3.** For t,r,s≥2, and 3≤n≤p in which p=t(r+s),

$$W_n(K_t\times K_{r,s})= p\binom{p-1}{n-1}+ rt[\binom{p-s-t}{n-1}+\binom{rt-t-r+1}{n-1}]$$
$$+st[\binom{p-r-t}{n-1}+\binom{ts-t-r+1}{n-1}] .$$

**Proof.** The n-Wiener index is obtained from $W_n(K_t \times K_{r,s};x)$ by taking the derivative with respect to x, and then put x=1, and simplified the expression.■

## 5. The Cartesian Product of a Complete Graph and a Wheel

Let $W_r$ be a wheel of order $r \geq 4$ and let its center be denoted by $v_0$ and its other vertices be $v_1, v_2, \ldots, v_{r-1}$. Moreover, let $V(K_t)=\{u_1, u_2, \ldots, u_t\}$. The order of $K_t \times W_r$ is p=rt, and in $K_t \times W_r$

$$deg(u_i,v_j)=t+2, \text{ for } 1\leq i\leq t,\ 1\leq j\leq r-1,$$
$$deg(u_i,v_0)=t+r-2.$$

One can easily see that in $K_t \times W_r$

$$d((u_i,v_0),(u_j,v_h))=2, \text{ for } i\neq j,\ h\neq 0,$$
$$d((u_i,v_h),(u_j,v_m))=3, \text{ for } i\neq j,\ h\neq m,\ h,m\neq 0,$$

because $(u_i,v_h)$, $(u_j,v_h)$, $(u_j,v_0)$, $(u_j,v_m)$ is a shortest $(u_i,v_h)$-$(u_j,v_m)$ when $v_hv_m \notin W_r$.Thus,

$$\text{diam } K_t\times W_r=3, \text{ when } r\geq 5.$$

Thus, for $r\geq 5$, $t\geq 2$,

$$\text{diam}_n\, K_t\times W_r\leq 3.$$

Since for each vertex $(u_i,v_h)$, $1\leq i\leq t$, $h\neq 0$ there are (t-1)(r-4) vertices of distance 3 from $(u_i,v_h)$, and $deg(u_i,v_h)=t+2$, then we have the following result.

**Proposition 5.1.** For $t\geq 2$, $r\geq 5$, the n-diameter of $K_t\times W_r$ is given by

$$\text{diam}_n\, K_t\times W_r= \begin{cases} 3, & \text{for } 2\leq n\leq 1+(t-1)(r-4), \\ 2, & \text{for } 2+(t-1)(r-4)\leq n\leq p-t-2, \\ 1, & \text{for } p-t-1\leq n\leq p. \end{cases} \qquad ■$$

The following theorem gives us the n-Wiener polynomial of $K_t\times W_r$.

**Theorem 5.2.** For $t\geq 2$, $r\geq 5$, $3\leq n\leq p$, p=tr

$$W_n(K_t\times W_r;x)= p\binom{p-1}{n-2}+\left[p\binom{p-1}{n-1}-t(r-1)\binom{p-t-3}{n-1}-t\binom{p-r-t+1}{n-1}\right]x$$
$$+\left[t(r-1)\binom{p-t-3}{n-1}+t\binom{p-t-r+1}{n-1}-t(r-1)\binom{p-r-4t+4}{n-1}\right]x^2$$
$$+t(r-1)\binom{p-r-4t+4}{n-1}x^3.$$

**Proof.** The coefficients of $x^0$ and x are obtained using (1.8) and (1.9). To obtain the coefficient of $x^3$, we notice that for any $(u_i,v_0)$, $1\leq i\leq t$ and every (n-1)-set of vertices S, $d_n((u_i,v_0),S)\leq 2$. But for every vertex $(u_i,v_j)$, $1\leq i\leq t$, $1\leq j\leq r-1$, there are (t-1)(r-4) vertices each of distance 3 from $(u_i,v_j)$.

Therefore, there are $\binom{p-r-4t+4}{n-1}$ sets S, |S|=n-1, such that $d_n((u_i,v_j),S)=3$. Thus,

$$C_n(K_t\times W_r,3)=t(r-1)\binom{p-r-4t+4}{n-1}.$$

We obtain $C_n(K_t\times W_r,2)$ by using (1.8). Hence, the proof. ■

**Corollary 3.4.3.** For t≥2, r≥5 and 3≤n≤rt,

$$W_n(K_t\times W_r)=p\binom{p-1}{n-2}+t(r-1)\binom{p-t-3}{n-1}+t\binom{p-r-t+1}{n-1}+t(r-1)\binom{p-r-4t+4}{n-1}$$

**Proof.** The proof follows from Theorem 5.2 and the fact

$$W_n(K_t\times W_r)=\acute{W}_n(K_t\times W_{r;1}).$$ ■

## 6. The Cartesian Product of a Path and a Complete Graph

Let $P_r$, r≥2 be a path graph of order r and

$$P_r: v_1,v_2,\ldots,v_r,$$

and let

$$V(K_t)=\{u_1,u_2,\ldots,u_t\},\ t\geq 3.$$

The Cartesian product $K_t\times P_r$ is shown in Fig. 6.1. The following proposition determines the n-diameter of $K_t\times P_r$ .

**Proposition 6.1.** For r≥2, t≥3, 2≤n≤rt,

$$\operatorname{diam}_n K_t\times P_r=r+1-\lceil n/t\rceil.$$

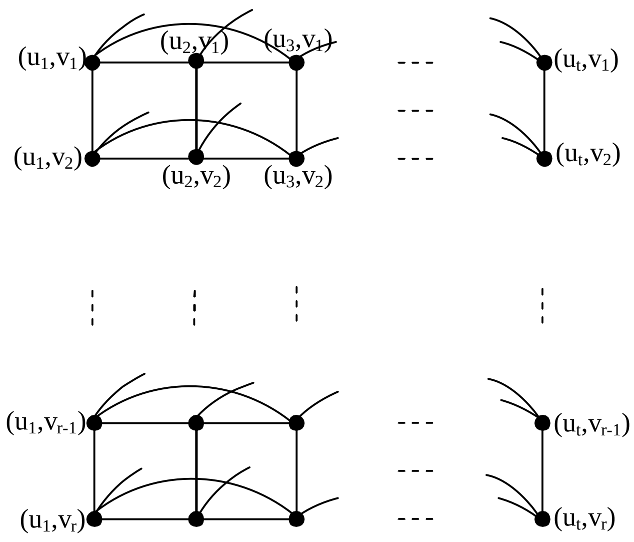


**Fig.6.1. The graph** $K_t\times P_r$

**Proof.** From Fig. 6.1, we notice that $d_n((u,v),S)$, $|S|=n-1$ has maximum value when $(u,v)$ is one of the vertices in $A_1 \cup A_r$, where

$$A_i=\{(u_j,v_i): j=1,2,\ldots,t\},$$

and S is the (n-1)-set of vertices farthest from $(u,v)$ in $K_t \times P_r$. Thus, we may take the vertex $(u_1,v_r)$, and S consisting of vertices of $A_1,A_2,\ldots,A_i$ and some vertices of $A_{i+1}-\{(u_1,v_{i+1})\}$ when

$$it \le n-1 \le t(i+1)-1;$$

and when

$$2 \le n \le t, \text{ then } S \subseteq A_1-\{(u_1,v_1)\}.$$

In the last case,

$$diam_n\, K_t \times P_r = r\ ;$$

and in general case of n,

$$diam_n\, K_t \times P_r = r-i,\ \ it+1 \le n \le (i+1)t.$$

One can easily see that

$$i=\lceil n/t \rceil -1.$$

Hence, in any case of the value of n,

$$diam_n\, K_t \times P_r = r+1-\lceil n/t \rceil .\ \blacksquare$$

Now, we obtain the n-Wiener polynomial of $K_t \times P_r$ in the following two theorems.

**Theorem 6.2.** Let $r=2s$, $s \ge 1$, $t \ge 3$ and $3 \le n \le rt$. Then

$$W_n(K_t \times P_r;x)=\sum_{k=0}^{d_n} C_n(K_t \times P_r,k)x^k,$$

where

$$C_n(K_t \times P_r,0)=rt\binom{rt-1}{n-2},$$

$$C_n(K_t \times P_r,1)=rt\binom{rt-1}{n-1}-2t\binom{rt-t-1}{n-1}-t(r-2)\binom{rt-t-2}{n-1},$$

for $2 \le k \le s$

$$C_n(K_t \times P_r,k)=2t[2\sum_{i=1}^{k-1}\{\binom{a+t-ti}{n-1}-\binom{a-it}{n-1}\}+2\{\binom{a+2t-tk-1}{n-1}-\binom{a-tk}{n-1}\}$$
$$+(s-k)\{\binom{a+2t-tk-1}{n-1}-\binom{a-tk-1}{n-1}\}],$$

for $k \ge s+1$

$$C_n(K_t \times P_r,k)=2t\sum_{i=1}^{s}\ \{\binom{a+t-ti}{n-1}-\binom{a-ti}{n-1}\},$$

in which

$$\alpha = p - t(k-1) - 1.$$

**Proof.** $C_n(K_t \times P_r, 0)$ and $C_n(K_t \times P_r, 1)$ are obtained from (1.8) and (1.9). For $2 \le k \le \delta_n$ we shall consider three cases for the values of k.

(1) If $2 \le k < s$, then for each $1 \le i \le s$ the number of vertices of distance k from any vertex, say $(u_j, v_i)$, of $A_i$ is t and the number of vertices of distance more than k from $(u_j, v_i)$ is $p-t(i+k-1)-1=\alpha-ti$ when $1 \le i \le k-1$ which gives us

$$a = \sum_{i=1}^{k-1}\sum_{j=1}^{n-1}\binom{t}{j}\binom{a-ti}{n-1-j} \qquad \ldots(6.1)$$

If i=k, then there are 2t-1 vertices of distance k from $(u_j, v_i)$, and there are p-t(2k-1)-1 vertices of distance more than k. This gives us

$$b = \sum_{j=1}^{n-1}\binom{2t-1}{j}\binom{p-2kt+t-1}{n-1-j}$$

$$= \binom{p-2kt+3t-2}{n-1} - \binom{p-2kt+t-1}{n-1}$$

$$= \binom{a+2t-tk-1}{n-1} - \binom{a-kt}{n-1}. \qquad \ldots(6.2)$$

If $k+1 \le i \le s$, then there are 2t vertices of distance k from $(u_j, v_i)$ and there are p-t(2k-1)-2 vertices of distance more than k. This gives us

$$c = \sum_{i=k+1}^{s}\sum_{j=1}^{n-1}\binom{2t}{j}\binom{p-2kt+t-2}{n-1-j}$$

$$=(s-k)[\binom{p-2kt+3t-2}{n-1} - \binom{p-2kt+t-2}{n-1}].$$

$$=(s-k)[\binom{a+2t-tk-1}{n-1} - \binom{a-kt-1}{n-1}]. \qquad \ldots(6.3)$$

Since r=2s and each $A_i$ consists of t vertices,

$$C_n(K_t \times P_r, k) = 2t(a+b+c) \text{ when } 2 \le k < s.$$

(2) If k=s, then using the same reasoning as in case (1) we find that (6.1) and (6.2) are true for this case, and (6.3) does not hold. Thus,

$$C_n(K_t \times P_r, k) = 2t(a+b) \text{ when } k = s.$$

(3) If $k \ge s+1$, then it is clear that both (6.2) and (6.3) do not hold. Thus,

$$C_n(K_t \times P_r, k) == 2ta \text{ when } k \ge s+1.$$

Substituting a, b and c, we get the required results. ■

**Theorem 6.3.** Let $r=2s+1$, $s\geq 1$, $t\geq 3$ and $3\leq n\leq rt$.
Then

$$W_n(K_t\times P_r;x)=\sum_{k=0}^{d_n} C_n(K_t\times P_r,k)x^k,$$

where

$$C_n(K_t\times P_r,0)=rt\binom{rt-1}{n-2},$$

$$C_n(K_t\times P_r,1)=rt\binom{rt-1}{n-1}-2t\binom{rt-t-1}{n-1}-t(r-2)\binom{rt-t-2}{n-1},$$

for $2\leq k\leq s$

$$C_n(K_t\times P_r,k)=4t[\sum_{i=1}^{k-1}\{\binom{a+t-ti}{n-1}-\binom{a-it}{n-1}\}+\binom{a+2t-tk-1}{n-1}-\binom{a-tk}{n-1}]$$
$$+t(r-2k)\{\binom{a+2t-tk-1}{n-1}-\binom{a-tk-1}{n-1}\},$$

for $k=s+1$,

$$C_n(K_t\times P_r,k)=2t\sum_{i=1}^{s}\{\binom{a+t-ti}{n-1}-\binom{a-ti}{n-1}\}+t\binom{2t-2}{n-1}$$

for $s+1<k\leq\delta_n$,

$$C_n(K_t\times P_r,k)=2t\sum_{i=1}^{s}\{\binom{a+t-ti}{n-1}-\binom{a-ti}{n-1}\},$$

in which

$$\alpha=p-t(k-1)-1.$$

**Proof.** The proof of $C_n(K_t\times P_r,k)$ for $k\neq s+1$ is similar to that for even r given in Theorem 6.2. For $k=s+1$ we add the number of pairs $((u_j,v_{s+1}),S)$ of n-distance s+1, which equals $\binom{2t-2}{n-1}$ for each $1\leq j\leq t$. ■

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