On The Elliptic Variational Inequality Of The First Kind For The Elasto-Plastic Torsion Problem

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Abstract

The elliptic variational inequality of the first kind for the " Elasto- Plastic torsion problem" is considered. This elliptic variational inequality is related to second order partial differential operator.The physical and mathematical interpretation and some properties of the solution are given.

الخلاصة

لقد اعتبرنا المتباينة التغايرية الناقصية من النوع الاول لمسألة الطي، هذه المتباينة لها علاقة بالمؤثر التفاضلي الجزئي من الرتبة الثانية ، التفسير الرياضي والفيزيائي وبعض خواص الحل قد اعطيت .

1. Introduction:

An important and very useful class of non- linear problems arising from mechanics, physics etc. consists of the so- called variational inequalities. In this paper we shall restrict our attention to the study of the existence, uniqueness, and properties of the solutions of elliptic variational inequality (EVI) of the first kind .

1-1: Notations :

V : real Hilbert space with scalar product (. , .) and associated norm $\| . \|$

V^* : The dual space of V .

a (. , .): $VxV \rightarrow$ IR is abilinear, continuous and $V-$ elliptic mapping on VxV .

A bilinear form a (. , .) is said to be $V-$ elliptic if there exists a positive constant α such that a $(v,v) \geq \alpha\|v\|^2$, $\forall v \in V$.

In general we do not assume a (. , .) to be symmetric , since in some applications non-symmetric bilinear forms may occur naturally (Lions , 1967).

L: $V \rightarrow$ IR continuous, linear functional.

K: is a closed, convex, non- empty subset of V .

j(.) : $V \rightarrow \overline{IR} = IR \cup \{\infty\}$ is a convex, lower semi-continuous (L.S.C.) and proper functional.

(j(.) is proper if $<$ j(v)$> \infty \forall v \in V$ and $j \neq \infty$)

1.2: Elliptic Variationd Inequality of First kind (EVI)

To find $u \in V$ such that u is a solution of the problem:

$$p_1 \cdots\cdots \begin{cases} a(u, v-u) \geq L(v-u), \forall v \in k \\ u \in k \end{cases}$$

1.3: Existence and Uniqueness Results for EVI of First kind

1.3.1: A Theorem of Existence and Uniqueness (Lions , 1967 **;** Chipot *et al*., 1987).

The problem p_1 has one and only one solution.

Proof: 1- Uniqueness

Let u_1 and u_2 be solutions of (p_1). We have then:

$a(u_1, v-u_1) \geq L(v-u_1) \qquad \forall v \in k, u_1 \in k \cdots\cdots\cdots (1)$

$a(u_2, v-u_2) \geq L(v-u_2) \qquad \forall v \in k, u_2 \in k \qquad \cdots\cdots\cdots(2)$

putting u_2 for v in (1) and u_1 for v in (2) and adding we get, by using the $V-$ ellipticity of a (.,.).

$$\alpha\|u_2 - u_1\|^2 \leq a(u_2 - u_1, u_2 - u_1) \leq 0$$

Which implies $u_1 = u_2$, since $\alpha > 0$

2- Existence: we will reduce the problem (p_1) to a fixed point problem . By the Riesz representation theorem for Hilbert spaces there exist .

$A \in \lambda(V,V), (A = A^t \ if \ a(.,.)$ is symmetric and $l \in V$ such that:-

$(Au, v) = a(u, v) \qquad \forall u, v \in V$

and $L(v) = (\ell, v) \qquad \forall v \in V$ ……….. (3)

Then the problem (ρ_1) is equivalent to finding $u \in V$ such that:-

$$(4)\ldots\ldots\begin{cases}(u - \rho(Au - l) - u, v - u) \leq o & \forall v \in k \\ u \in k & ,\rho > 0\end{cases}$$

This is equivalent to finding u such that:

$$u = p_k(u - p(Au - \ell), for \quad some \quad \rho > 0 \qquad \ldots\ldots\ldots(5)$$

Where p_k denotes the projection operator from V to k in the $\|.\|$ norm.

Consider the map $W_\rho : V \to V$ defined by:

$$W_\rho(v) = p_k(v - \rho(Av - \ell) \qquad \ldots\ldots\ldots(6)$$

let $v_1, v_2 \in V$, then since p_k is a contraction we have

$$\{\|w_\rho(v_1) - w_\rho(v_2)\|^2 \leq \|v_2 - v_1\|^2 + \rho^2\|A(v_2 - v_1)\|^2$$
$$-2\rho a(v_2 - v_1, v_2 - v_1)$$

Hence we have

$$\|w_\rho(v_1) - w_\rho(v_2)\|^2 \leq (1 - 2\rho\alpha + \rho^2\|A\|^2)\|v_2 - v_1\|^2 \qquad \ldots\ldots\ldots(7)$$

Thus w_ρ is a strict contraction mapping if $0 < \rho < \frac{2\alpha}{\|A\|^2}$. By taking ρ in this range we have a unique solution for the fixed point problem which implies the existence of a solution for (p_1).

2- An Example of EVI of the first kind " The Elasto – plastic Torsion Problem"

2.1: Notations

Ω : a bounded domain in IR^2

$\Gamma : \partial\Omega$

X={ x_1,x_2} a generic point of Ω

$$\nabla = \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\}$$

$C^m(\overline{\Omega})$: space of m- times continuously differentiable real valued functions for which all the derivatives up to order m are continuous in $\overline{\Omega}$

$$C_0^m(\Omega) = \{ v \in C^m(\overline{\Omega}) : \sup p(v) \, is\, a\, compact\, subset\, of\, \Omega \}$$

$$\|v\|_{m,p,\Omega} = \sum_{|\alpha| \le m} \|D^\alpha v\|_{L^p(\Omega)} \, for\, v \in c^m(\overline{\Omega}) \, where\, \alpha = (\alpha_1, \alpha_2); \alpha_1, \alpha_2 \; non-negative$$

$$\text{int}egers, |\alpha| = \alpha_1 + \alpha_2 \; and \; D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$$

$w^{m,p}(\Omega)$: completion of $C^m(\overline{\Omega})$ in the norm defined above .

$w_o^{m,p}(\Omega)$: completion of $C_o^m(\Omega)$ in the above norm.

$H^m(\Omega) = w^{m,2}(\Omega)$

$H_0^m(\Omega) = w_0^{m,2}(\Omega)$

2.2:The mathematical Interpretation of the problem

Let Ω be a bounded domain of IR^2 with a smooth boundary Γ, we consider the following EVI of the first kind:

$$(8)....\begin{cases} a(u, v-u) \ge L(v-u) & \forall v \in k \\ u \in k \end{cases}$$

where

$$k = \{ v \in H_0^1(\Omega) : |\nabla_v| \le 1 \quad a.e.\, in \Omega \} \qquad ....(9)$$

$$V = H_0^1(\Omega) = \{ v \in H^1(\Omega) : v|\Gamma = trace\, of\; v\; on\; \Gamma = 0$$

$$a(u,v) = \int_\Omega \nabla u . \nabla v, such\; that:$$

$$\nabla u . \nabla v = \frac{\partial u}{\partial x_1} . \frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2} . \frac{\partial v}{\partial x_2}$$

and L (v)= $< f, v >$ for $f \in V^* = H^{-1}(\Omega)$ and $v \in V$.

2.3: The physical Interpretation of the Problem

Let us consider on infinitely long cylindrical bar of cross-section Ω where Ω is simply connected. Assume that this bar is made up of an isotropic, elastic, perfectly plastic material whose plasticity yield is given by the von Mises criterion. (Koiter , 1987 ; Duvaut *et al*., 197]).

Starting from a zero stress initial state, an increasing torsion moment is applied to the bar. The torsion is charcterised by c, which is defined as the torsion angle per unit length (William , 2004). Then for all c, it follows from the Haar- karman principle that the determination of the stress field is equivalent to the solution of the following variational problem:-

$$\underset{v\in k}{Min}\ \frac{1}{2}\int_{\Omega}|\nabla v|^{2}dx-c\int_{\Omega}vdx$$

..........(10)

This is a particular case of (2.2) with

$L(v)=c\int_{\Omega}vdx$ ....(11)

3. Existence and Uniqueness Results of the " Elasto-plastic Torsion Problem"

In order to apply theorem (1.3.1) , we only have to verify that k is a non- empty , closed, convex , subset of V .

1- K is non- empty because $0\in K$, and the convexity of K is obvious.

2- To prove that K is closed , consider a sequence $\{v_n\}$ in k such that $v_n\to v$ strongly in V .

Then there exists a subsequence $|v_{n_i}|$ such that. $\lim_{i\to\infty}\ \nabla v_{ni}=\nabla v\quad a.e$

$\sin ce\ |\nabla v_{n_i}|\le 1\ a.e.\quad we\ get\ |\nabla v|\le\quad a.e.$

Therefore $v\in K$. Hence K is closed.

4. Regularity Properties and exact solutions

4.1 Regularity results (Chipot *et al.*, 1987)

4.1.1 Theorem

Let u be a solution of (8) and $L(v)=\int_{\Omega}fvdx$

1- Let Ω be a bounded convex domain of IR^2 with Γ lipschitz continuous and $f\in L^p(\Omega)$ with 1<p<∞, Then we have

$u\in w^{2,p}(\Omega)$

2- If Ω is a bounded domain of IR^2 with a smooth boundary, Γ $if\in L^p(\Omega)$ with $1<p<\infty$ then $u\in w^{2,p}(\Omega)$.

4.2 Exact solutions

In this section we are going to give an example of problems (8) for which exact solutions are known .

Example :- we take Ω={ x:0<x<1} and

$$L(v)=c\int_0^1 vdx\ with\ c>0.$$

Then the explicit form of (8) is

$$\left.\begin{aligned}&\int_0^1 u'(v'-u')dx\ \ge\ c\int_0^1 (v-u)dx\quad \forall v\in k\\&u\in k\end{aligned}\right\}\quad ....(12)$$

where $k = \{v \in H_0^1(\Omega) : |v'| \le 1 \quad a.e. \quad on\,\Omega\} and\ v' = \dfrac{dv}{dx}$

The exact solution of (12) is given by

$$u(x) = \frac{c}{2}x(1-x) \quad \forall x,\ if\ c \le 2 \qquad ....(13)$$

$if\ c > 2$

$$u(x) = \begin{cases} x\ if\ 0 \le x \le \frac{1}{2} - \frac{1}{c} \\ \frac{c}{2}\left[x(1-x) - (\frac{1}{2} - \frac{1}{c})^2\right] & if\ \frac{1}{2} - \frac{1}{c} \le x \le \frac{1}{2} + \frac{1}{c} \\ 1-x \qquad if\ \frac{1}{2} + \frac{1}{c} \le x \le 1 \end{cases}$$

5. An equivalent variational formulation (Lions , 1967)

If Ω is a bounded domain of IR^2 with a smooth boundary Γ and if

$$L(v) = c\int_\Omega v(x)dx \qquad (c > 0\ for\ ins\tan ce),$$

Then the solution of (8) is also a solution of

$$\begin{cases} a(u, v-u) \ge c\int (v-u)dx \quad \forall\ v \in \hat{k}, \\ u \in \hat{k} = \{v \in H_0^1(\Omega), |v(x)| \le d(x,\Gamma)\} \end{cases} \qquad ....(14)$$

Since a(. , .) is symmetric , (14) is also equivalent to

$$\begin{cases} J(u) \le J(v) \qquad \forall v \in \hat{k}, \\ u \in \hat{k} \end{cases} \qquad ....(15)$$

with $J(v) = \dfrac{1}{2}a(v,v) - c\displaystyle\int_\Omega v(x)dx$

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