Rings in which Every Simple Right R-Module is Flat

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Received on: 23/9/2002 Accepted on: 5/11/2002

ABSTRACT

The objective of this paper is to initiate the study of rings in which each simple right R-module is flat, such rings will be called right SF-rings. Some important properties of right SF-rings are obtained. Among other results we prove that: If R is a semi prime ERT right SF-ring with zero socle, then R is a strongly regular ring.
1. Introduction:

Throughout this paper, R denotes associative ring with identity and all modules are unitary. J(R) and Y(R) denote the Jacobson radical and the singular right ideal of R, respectively. For any nonempty subset X of a ring R, the right (resp. left) annihilater of X will be denoted by r(X) (resp. l(X)). Recall that:

1. A ring R is ERT if every essential right ideal of R is a -sided,[6]
2. R is said to be von Nuemann regular (or just regular) if,
   \[ a \in aRa, \text{for every } a \in R, \text{ and } R \text{ is called strongly regular if } a \in a^2 R. \]

In [3] Ming asked the following question:

Is a semi prime right SF-ring, all of whose essential right ideals are two-sided von Neumann regular?

In this paper, we give conditions for a semi prime right SF-ring all of whose essential right ideals are two sided to be von Neumann regular.

2. Basic Properties:

Following [5], a ring R is called a right (left) SF-ring, if every simple right (left) R-module is flat.

The following lemma which is due to [7], plays a central role in several of our proofs: Lemma 2.1:
Let I be a right ideal of R. Then \( R/I \) is a flat R-module if and only if for each \( a \in I \), there exists \( b \in I \) such that \( a = ba \).

We shall begin with the following result:

**Proposition 2.2:**

If \( R \) is a right SF-ring. Then

1. Any reduced principal right ideal of \( R \) is a direct summand.
2. Every left or right \( R \)-module is divisible.

**Proof (1):**

Let \( I = aR \) be a reduced principal right ideal of \( R \) and let \( aR + r(a) \neq R \). Then there exists a maximal right ideal \( M \) of \( R \) containing \( aR + r(a) \).

Now, since \( R/M \) is flat, then \( a = ba \), for some \( b \) in \( R \). Whence \( l - b \in l(a) = r(a) \subseteq M \). Yielding \( l \in M \) which contradicts \( M * R \). In particular

\[ ar + c = l, \text{ for some } r \in R \text{ and } c \in r(a), \text{ whence } a^2 r = a. \]

If we set \( d = ar^2 \in l \), then \( a = a^2 d \). Clearly, \((a - ada)^2 = 0\) implies \( a = ada \) and hence \( I = eR \), where \( e = ad \), is idempotent element. Thus \( I \) is a direct summand.

**Proof (2):**

It is sufficient to prove that any non-zero divisor \( c \) of \( R \) is invertible. For then, if \( dc = cd = l \), any right \( R \)-module \( M \) satisfies \( M = Mdc \subseteq Mc \subseteq M \), whence \( M = Mc \) (similarly, any left
R-module is divisible). Suppose that cR≠R. Let K be a maximal right ideal containing cR. Since R/K is flat, there exist ueK, such that c=uc. Now, r(c) =1 (c )=0 implies u=l, contradicting K≠R. This proves that cd=l for some d∈R and hence dc=l.

Proposition 2.3:

Let R be a right SF-ring. Then either r(M)=0 or M is a direct summand.

Proof:

Suppose that r(M)≠0 and let b∈M∩r(M). Then b∈M and Mb=0. Since R/M is flat then there exists a∈M such that b=ab. Now b=ab∈Mb=0, so b=0. Thus M∩r(M)=0, this means that M not can be essential and hence M is a direct summand. Therefore r(M)⊕M=R.

3. The Connection Between SF-Rings and Other Rings:

In this section we study the connection between SF-rings, biregular rings and strongly regular rings.

Recall that the right (left) socle of a ring R is defined to be the sum of all minimal right (left) ideals of R. It is well know that in a semi prime ring R, the right and left socles of R coincide, which will be denoted by socR[8].

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Following [4], a ring \( R \) is biregular if \( RaR \) is generated by a central idempotent for each \( a \in R \).

**Theorem 3.1:**

Let \( R \) be an ERT SF-ring with right zero socle and for every \( a \in R \), \( RaR \) is a principal right of \( R \). Then \( R \) is biregular.

Proof:

For any \( a \in R \), set \( M=RaR+\text{l}(RaR). \) Since \( M \) is a maximal right ideal, then \( M \) is a direct summand or essential. If \( M \) is a direct summand of \( R \), then its complement is a minimal right ideal. This implies that \( R \) has a no-zero socle, which is a contradiction. So every maximal right ideal is essential and hence two-sided. By hypothesis \( R/M \) is flat. Also \( RaR=bR \) for some \( b \in R \). Since \( b \in M \), \( b=db \) for some \( d \in M \). Then \( 1-d \in l(b)cM \) which yields \( 1 \in M \). Therefore \( 1=bc+v, c \in R, v \in l(b) \) this implies that \( b=bcb \). Therefore \( RaR=bR=eR \) where \( e=bc \) is idempotent. \( R \) is therefore semi-prime and hence \( e \) is central in \( R \). Thus \( R \) is biregular.

**Theorem 3.2:**

If \( R \) is a reduced ring and every maximal right ideal of \( R \) is either a right annihilator or flat, then \( R \) is strongly regular.
Proof:

Let $b \in R$. We claim first $bR + r(b) = R$. If not, there exists a maximal right ideal $L$ containing $bR + r(b)$. In case $R/L$ is flat, since $b \in L$, there exists $c \in L$ such that $b = cb$. Then $l - c \in l(b) = r(b)cL$, whence it follows $1 \in L$, a contradiction. On the other hand, in case $L = r(t)$ with some $0 \neq t \in R$, we have $t \in !(bR + r(b)) \subseteq l(b) = r(b) \subseteq L = r(t)$. Then $t^2 = 0$, a contradiction. Therefore let $bR + r(b) = R$, and hence $R$, is strongly regular.

Now, we give under what condition the answer of the question of ring is affirmative.

Proposition 3.2:

Let $R$ be a semi-prime ERT right SF-ring with zero socle. Then $R$ is strongly regular.

Proof:

Let $M$ be a maximal right ideal of $R$. Then $M$ is either a direct summand of $R$ or an essential right ideal of $R$. If $M$ is a direct summand of $R$, then its complement is a minimal right ideal. This implies that $R$ has a nonzero socle, which is a contradiction. So every maximal right ideal is essential and hence two-sided.

Since $R$ is semi-prime ERT, by applying [2,Lemma 2.1], we see that $R$ is right non-singular, and $J(R) = 0$. So $R$ is isomorphic to a sub direct sum of division rings, which implies that $R$ has no

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non-zero nilpotent elements. Therefore $R$ is strongly regular [1, proposition 1].

**Theorem 3.3:**

Let $R$ be a prime ERT and right SF-ring. Then $R$ has non-zero socle.

**Proof:**

Let $R$ be a prime ERT and right SF-ring. If $\text{Soc} R = 0$. By the above Theorem (3.2), $R$ is strongly regular. Hence $R$ is a division ring, and $\text{Soc} R = R$ contradicting our assumption. Therefore $\text{Soc} R \neq (0)$.

**REFERENCES**


