Detour Hosoya Polynomials of Some Compound Graphs

Herish O. Abdullah  Gashaw A. Muhammed-Saleh

College of Science
University of Salahaddin

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ABSTRACT

In this paper we will introduce a new graph distance based polynomial; Detour Hosoya polynomials of graphs $H^*(G;x)$. The Detour Hosoya polynomials $H^*(G;x)$ for some special graphs such as paths and cycles are obtained. Moreover the Detour Hosoya polynomials $H^*(G_1\bullet G_2;x), H^*(G_1\cdot G_2;x)$ and $H^*(G_1\cup G_2;x)$ are obtained.

Keywords: Detour distance, compound graphs, Hosoya polynomials.

1. Introduction

The concept of Hosoya polynomial was first put forward in 1988 by Hosoya [1]. Several authors, such as [1], [2], [3], [4], [5], [6], [7], [8], [13] and [15] had obtained Hosoya polynomials for special graphs, graphs having some kind of regularity and for compound graphs obtained by using some well-known binary operations in graph theory.

In this paper, we consider finite connected graphs without loops or multiple edges. For undefined concepts and notations see [9] and [12].

Ordinarily, when we wish to proceed from a point $A$ to a point $B$ we take a route which involves the least distance. We have all been faced with detour sign which require us to take a route from $A$ to $B$ that involves a greater distance. In any such detour route from $A$ to $B$ we assume that there is no possible shortcut along the route, for otherwise this should have been part of the route initially. When one is driving along such a detour, it sometimes seems that we are using the longest route possible from $A$ to $B$
(again subject to the “no shortcut” condition). In this paper we investigate longest detour routes in graphs.

The distance \( d(u,v) \) between two vertices \( u \) and \( v \) in a connected graph \( G \) is the length of a shortest \( u-v \) path in \( G \). For a nonempty set \( S \) of vertices of \( G \), the subgraph \( <S> \) of \( G \) induced by \( S \) as its vertex set while an edge of \( G \) belongs to \( <S> \) if it joins two vertices of \( S \). If \( P \) is a \( u-v \) path of length \( d(u,v) \), then the subgraph \( <V(P)> \) induced by the vertices of \( P \) is \( P \) itself. This observation suggests the following concept.

The detour distance \( d^*(u,v) \) between \( u \) and \( v \) in \( G \) is the length of a longest induced \( u-v \) path, that is a longest \( u-v \) path \( P \) for which \( <V(P)> \geq P \). An induced \( u-v \) path of length \( d^*(u,v) \) is called a detour path [10].

Observe that \( d^*(u,v) \geq d(u,v) \) for all vertices \( u \) and \( v \) of \( G \) and that \( d^*(u,v) = d(u,v) = 1 \) if and only if \( u \) and \( v \) are adjacent. Also, note that \( d^*(u,v) = d^*(v,u) \) for all vertices \( u \) and \( v \) of \( G \). Therefore the detour distance is symmetric. However, the triangle inequality does not hold in general. Consider the wheel \( W_p \) of order \( p \geq 6 \) with center at the vertex \( w \); then: \( d^*(u,v) = p - 3 > 2 = d^*(u,w) + d^*(w,v), \) for every two vertices \( u \) and \( v \) of \( W_p \), \( u,v \neq w \), that are both adjacent to a common vertex \( x \neq w \).

Therefore, in general, the detour distance is not a metric on the vertex set of \( G[10] \).

The detour eccentricity \( e^*(v) \) of a vertex \( v \) is defined by \( e^*(v) = \max\{d^*(v,w) : w \in V(G)\} \). The detour eccentricity set \( e^*(G) \) of a connected graph \( G \) is the set consisting of all detour eccentricities of \( G \), that is \( e^*(G) = \{e^*(v) : v \in V(G)\} \). The detour radius \( rad^*(G) \) of \( G \) is the minimum detour eccentricity, while the detour diameter \( diam^*(G) \) of \( G \) is the maximum detour eccentricity.

For completeness we define \( d^*(u,v) = 0 \) if and only \( u = v \).

A connected graph \( G \) is called a detour graph if \( d^*(u,v) = d(u,v) \) for all vertices \( u \) and \( v \) of \( G \). No cycle of length 5 or more is a detour graph.
On the other hand, all trees and all complete graphs are detour graphs. If \( u \) and \( v \) are distinct vertices of a graph \( G \) such that \( d^*(u,v) = 1 \) or \( 2 \), then \( d(u,v) = d(u,v) \)[10], the converse is not true in general, that is if \( d(u,v) = 2 \), then \( d^*(u,v) \geq 2 \), as for the wheel \( W_p, \ p \geq 6 \).

The concept of Hosoya polynomial \( H(G;x) \) of a graph \( G \) was put forward by Hosoya[13], and defined as

\[
H(G;x) = \sum_{k=0}^{\delta(G)} C(G,k)x^k
\]

where \( C(G,k) \) is the number of pairs of vertices in \( G \) that are distance \( k \) apart, and \( \delta(G) \) is the diameter of the graph \( G \).

In this paper, the concept of **Hosoya polynomials of detour distance** of a connected graph \( G \) (or simply **detour Hosoya polynomial of a graph** \( G \)) has been defined by

\[
H^*(G;x) = \sum_{k=0}^{\delta^*(G)} C^*(G,k)x^k = \sum_{\{u,v\} \subseteq V(G)} x^{d^*(u,v)} \quad \cdots(1)
\]

in which \( C^*(G,k) \) is the number of pairs of vertices in \( G \) with detour distance \( k \), and \( \delta^*(G) \) is the detour diameter of \( G \).

It is clear that if \( G \) is a detour graph, then \( H^*(G;x) = H(G;x) \).

The sum \( W^*(G) \) of detour distances between all pairs of vertices of the graph \( G \) is known as the **Wiener index of detour distance** of the graph \( G \) (or simply **detour Wiener index** of the graph \( G \)), that is

\[
W^*(G) = \sum_{u,v} d^*(u,v),
\]

where the sum is taken over all unordered pairs \( \{u,v\} \) of distinct vertices in \( G \).

It is clear that

\[
W^*(G) = \frac{d}{dx} H^*(G;x) \bigg|_{x=1}.
\]

We illustrate these ideas in the following example.

**Example 1.1.** Let \( G \) be a graph of order \( p = 9 \), depicted in figure 1.1(a).

It is clear that

\[
e^*(v_1) = 5, \ e^*(v_2) = 4, \ e^*(v_3) = 4, \ e^*(v_4) = 3, \ e^*(v_5) = 4, \\
e^*(v_6) = 3, \ e^*(v_7) = 4, \ e^*(v_8) = 5 \text{ and } e^*(v_9) = 5.
\]
Hence
\[ e^*(G) = \{5, 4, 4, 3, 4, 3, 4, 5, 5\}, \quad \text{diam}^*(G) = 5 \quad \text{and} \quad \text{rad}^*(G) = 3. \]

A detour \( v_1 - v_9 \) path is given in Figure 1.1(b). Therefore \( d^*(v_1, v_9) = 5 \), and this gives us the maximum detour distance among all detour distances of pairs of vertices of \( V(G) \).

The path \( P' \) is not a detour \( v_1 - v_9 \) path, because \( \{V(P')\} \neq P' \) (see figures 1.1(c) and 1.1(d)).

By direct calculations, we get that
\[ C^*(G, 0) = p = 9, \quad C^*(G, 1) = 10, \quad C^*(G, 2) = 9, \]
\[ C^*(G, 3) = 9, \quad C^*(G, 4) = 6 \quad \text{and} \quad C^*(G, 5) = 2. \]

Hence, the detour Hosoya polynomial of \( G \) is
\[ H^*(G; x) = 9 + 10x + 9x^2 + 9x^3 + 6x^4 + 2x^5, \]
and
\[ W^*(G) = \frac{d}{dx}H^*(G; x) \bigg|_{x=1} = 89. \]

![Figure 1.1](image)

(a) The graph \( G \)
(b) The detour \( v_1 - v_9 \) path

(c) The path \( P' \)
(d) \( \{V(P')\} \)

In 1993, Gutman [8], established few additional properties of the respective graph polynomials. He obtained Hosoya polynomials of some special graphs and obtained formula for the Hosoya polynomials of some compound graphs, namely \( G_1 \cdot G_2 \) and \( G_1 : G_2 \) which are defined in the following: Let \( G_1 \) and \( G_2 \) be vertex-disjoint connected graphs, and let \( u \in V(G_1) \) and \( v \in V(G_2) \). Then, the graph \( G_1 \cdot G_2 \) is obtained from
\(G_1\) and \(G_2\) by identifying the two vertices \(u\) and \(v\). This means that \(G_1\) and \(G_2\) have exactly one vertex in common in the compound graph \(G_1 \cdot G_2\). The graph \(G_1 \cdot G_2\) is obtained from \(G_1\) and \(G_2\) by introducing a new edge joining the two vertices \(u\) and \(v\). In this paper, formulas for \(H^*(G_1 \cdot G_2; x)\) and \(H^*(G_1 \cdot G_2; x)\) in terms of the detour Hosoya polynomials of \(G_1\) and \(G_2\) will be obtained.

2. Detour Hosoya Polynomials of Some Special Graphs

Let \(P_n\), \(K_n\) and \(S_n\) denotes the path, complete and star graphs of \(n\) vertices respectively. It is known that [10] all trees and complete graphs are detour graphs. This leads us to the following result.

**Proposition 2.1**

\[
\begin{align*}
(a) \quad H^*(P_n; x) &= \sum_{k=0}^{n-1} (n-k)x^k. \\
(b) \quad H^*(K_n; x) &= n + \frac{1}{2}n(n-1)x. \\
(c) \quad H^*(S_n; x) &= n + (n-1)x + \left(\frac{n-1}{2}\right)x^2. 
\end{align*}
\]

**Proof.** Let \(C_p\) be a cycle of order \(p \geq 5\), then

\[
H^*(C_p; x) = \begin{cases} 
\sum_{k=\frac{p+1}{2}}^{p} x^k & \text{if } p \text{ is odd} \\
p(1 + x + \frac{1}{2}x^\frac{p}{2} + \sum_{k=\frac{p}{2}+1}^{p-2} x^k) & \text{if } p \text{ is even} 
\end{cases}
\]

**Proposition 2.2** Let \(C_p\) be a cycle of order \(p \geq 5\), then

\[
H^*(C_p; x) = \begin{cases} 
p(1 + x + \sum_{k=\frac{p+1}{2}}^{p} x^k) & \text{if } p \text{ is odd} \\
p(1 + x + \frac{1}{2}x^\frac{p}{2} + \sum_{k=\frac{p}{2}+1}^{p-2} x^k) & \text{if } p \text{ is even} 
\end{cases}
\]

**Proof.** Let \(u,v\) be any two distinct vertices of \(C_p\). We will consider the following cases:

1. If \(uv \in E(C_p)\) then \(d^*(u,v) = 1\) and \(C^*(G,1) = p\).
2. If \(uv \notin E(C_p)\), then \(d^*(u,v) = p - d(u,v)\), where \(d(u,v)\) denotes the ordinary distance.
We know that \[11\], for an odd \(p\), the ordinary Hosoya polynomial of \(C_p\) is given by \(H(C_p; x) = p + px + p \sum_{k=2}^{\frac{p-1}{2}} x^k\).

Hence
\[
H^*(C_p; x) = p + px + p \sum_{k=2}^{\frac{p-1}{2}} x^k
\]
or
\[
H^*(C_p; x) = p + px + p \sum_{k=\frac{p+1}{2}}^{p-2} x^k.
\]
Similarly, we prove the formula for the case when \(p\) is even. This completes the proof. ■

**Proposition 2.3** Let \(W_p\) be a wheel graph of \(p \geq 6\) vertices, then
\[
H^*(W_p; x) = p + 2(p-1)x + (p-1)\left\{ \begin{array}{ll}
\sum_{k=\frac{p}{2}}^{p-3} x^k, & \text{if } p \text{ is even} \\
\frac{1}{2}x^{\frac{p-1}{2}} + \sum_{k=\frac{p+1}{2}}^{p-3} x^k, & \text{if } p \text{ is odd}
\end{array} \right.
\]

**Proof.** For \(uv \in E(W_p)\), \(d_{W_p}^*(u, v) = d_{C_{p-1}}^*(u, v)\).

Hence, for \(k \geq 2\)
\[
C^*(W_p, k) = C^*(C_{p-1}, k).
\]
Thus,
\[
H^*(W_p; x) = 1 + (p-1)x + H^*(C_{p-1}, x).
\]
Now, using Proposition 2 we obtain the required result. ■

**Proposition 2.4** Let \(K_{t,s}\) be a complete bipartite graph with partite subsets of sizes \(t\) and \(s\), then
\[
H^*(K_{t,s}; x) = (t + s) + (ts)x + \left[ \binom{t}{2} + \binom{s}{2} \right] x^2.
\]

**Proof.** Obvious ■
The following result gives us the Wiener index of the detour distance of the special graphs \( P_n \), \( K_n \), \( S_n \), \( C_p \), \( W_p \) and \( K_{t,s} \).

**Proposition 2.5**

1. \( W^*(P_n) = \frac{1}{6} n(n^2 - 1) \).
2. \( W^*(K_n) = \frac{1}{2} n(n - 1) \).
3. \( W^*(S_n) = (n - 1)^2 \).
4. For \( p \geq 5 \), \( W^*(C_p) = \begin{cases} \frac{1}{3} p(3p^2 - 12p + 17), & \text{if } p \text{ is odd} \\ \frac{1}{3} p(3p^2 - 12p + 16), & \text{if } p \text{ is even} \end{cases} \)
5. For \( p \geq 6 \), \( W^*(W_p) = \begin{cases} \frac{1}{8} (p-1)(3p^2 - 18p + 39), & \text{if } p \text{ is odd} \\ \frac{1}{8} (p-1)(3p^2 - 18p + 40), & \text{if } p \text{ is even} \end{cases} \)
6. \( W^*(K_{t,s}) = ts + t(t - 1) + s(s - 1) \).

3. **Detour Hosoya Polynomials of Some Compound Graphs**

Let \( u \) be a vertex of a connected graph \( G \) of order \( p \). The number of pairs of vertices of \( G \) containing the vertex \( u \) such that \( d_G^*(u,v) = k \), \( \forall v \in V(G) \), will be denoted by \( C^*(u,G;k) \).

We define the polynomial

\[
H^*(u,G;x) = \sum_{k=0}^{\epsilon^*(u)} C^*(u,G;k)x^k \quad \text{ ...(2)}
\]

It is clear that

\[
H^*(G;x) = \frac{1}{2} \sum_{u \in V(G)} H^*(u,G;x) + \frac{1}{2} p \quad \text{ ...(3)}
\]

Let \( G_1 \) and \( G_2 \) be two disjoint connected graphs of orders \( p_1 \) and \( p_2 \) respectively. Moreover, let \( w \) be the vertex obtained by identifying the vertex \( u \) of \( G_1 \) with the vertex \( v \) of \( G_2 \) in order to construct the compound graph \( G_1 \cdot G_2 \). The compound graph \( G_1 : G_2 \) is obtained by introducing a new edge joining the vertex \( u \) of \( G_1 \) with the vertex \( v \) of \( G_2 \).

Now, we are ready to present formulas for \( H^*(G_1 \cdot G_2;x) \) and \( H^*(G_1 : G_2;x) \) in terms of \( H^*(G_1;x) \) and \( H^*(G_2;x) \).
Theorem 3.1 If $G_1$ and $G_2$ are disjoint connected graphs, then

$$H^*(G_1 \cdot G_2;x) = H^*(G_1;x) + H^*(G_2;x) + H^*(u,G_1;x). H^*(v,G_2;x) - H^*(u,G_1;x) - H^*(v,G_2;x).$$

Proof: Let $s, t$ be any two vertices of $G_1 \cdot G_2$ such that $d_{G_1 \cdot G_2}^*(s,t) = k$. We will consider the following cases:

1. If $s,t \in V(G_1)$, then $C^*(G_1 \cdot G_2;k) = C^*(G_1,k)$, which produces the polynomial $H^*(G_1;x)$.
2. If $s,t \in V(G_2)$, then $C^*(G_1 \cdot G_2;k) = C^*(G_2,k)$, which produces the polynomial $H^*(G_2;x)$.
3. $s \in V(G_1)$ and $t \in V(G_2)$: In this case, any longest induced $(s,t)$-path $P$ will contain the vertex $w$. If $P'$ is a longest $(s,w)$-path and $P''$ is a longest $(t,w)$-path with $\langle V(P') \rangle = P'$ and $\langle V(P'') \rangle = P''$, then

$V(P) = V(P') \cup V(P'')$, and $\langle V(P) \rangle = \langle V(P') \cup V(P'') \rangle$,

because no vertex of $P'$, other than $w$ is adjacent with a vertex of $P''$, other than $w$.

Therefore $P' \cdot P'' = \langle V(P) \rangle = P$.

Hence, $d_{G_1 \cdot G_2}^*(s,t) = d_{G_1}^*(s,w) + d_{G_2}^*(t,w)$.

This produces the polynomial $H^*(u,G_1;x). H^*(v,G_2;x)$. Notice that the polynomial $H^*(u,G_1;x)$ is counted twice in the Cases (1) and (3), and also $H^*(v,G_2;x)$ is counted twice in the Cases (2) and (3).

Now, adding the polynomials obtained from the cases (1), (2) and (3), we get the required result.

Theorem 3.2 If $G_1$ and $G_2$ are disjoint connected graphs, then

$$H^*(G_1 : G_2;x) = H^*(G_1;x) + H^*(G_2;x) + H^*(u,G_1;x). H^*(v,G_2;x).$$

Proof: Let $s, t$ be any two distinct vertices of the compound graph $G_1 : G_2$. We consider the following cases:

1. If $s,t \in V(G_1)$, then we get the polynomial $H^*(G_1;x)$.
2. If $s,t \in V(G_2)$, then we get the polynomial $H^*(G_2;x)$.

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(3) $s \in V(G_1)$ and $t \in V(G_2)$: In this case, any longest $(s,t)$-path will contains the edge $uv$, and as in the proof of Theorem 6(Case 3), this produces the polynomial
\[ x \cdot H^*(u,G_1;x) \cdot H^*(v,G_2;x) \]
Now, adding the polynomials obtained from the cases (1), (2) and (3), we get the required result.

**Definition 3.3** Let $G_1$ and $G_2$ be disjoint connected graphs of orders $p_1$ and $p_2$, respectively. Let $G_2^{(i)}$ be the $i^{th}$ copy of $G_2$. The **Corona** $G_1 \odot G_2$, is the graph[13] constructed from $G_1 \cup p_1G_2$ with additional edges $\bigcup_{i=1}^{p_1} \{v : u \in V(G_2^{(i)})\}$, as depicted in Fig. 3.1, in which $V(G_1) = \{v_1,v_2,...,v_{p_1}\}$.

It is clear that
\[ p(G_1 \odot G_2) = p_1(1 + p_2) = p , \]
and
\[ q(G_1 \odot G_2) = q(G_1) + p_1(p_2 + q(G_2)) = q . \]

![Fig. 3.1 The Corona $G_1 \odot G_2$](image)

The next theorem computes the detour Hosoya polynomial of the corona $G_1 \odot G_2$. 

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**Detour Hosoya Polynomials of Some Compound Graphs**
**Theorem 3.4** Let $G_1$ and $G_2$ be two disjoint connected graphs, then

$$H^*(G_1, G_2; x) = (1 + p_2 x)^2 H^*(G_1; x) + p_1 H^*(G_2; x) - p_1 p_2 x (1 + p_2 x).$$

**Proof.** Let $s, t$ be any two distinct vertices of $G_1$ and $G_2$. We will consider the following cases:

**Case 1.** If $s, t \in V(G_1)$, then we get the polynomial $H^*(G_1; x)$.

**Case 2.** If $s, t \in V(G_2^{(i)})$, for $i = 1, 2, \ldots, p_1$, then we get the polynomial $p_1 H^*(G_2; x)$.

**Case 3.** $s \in V_2^{(i)}$ and $t = v_j$ (or $s = v_i$ and $t \in V_2^{(j)}$) for $i, j = 1, 2, \ldots, p_1$, then

(i) If $i = j$, then we get the polynomial $p_1 p_2 x$.

(ii) If $i \neq j$, then we get the polynomial $2 p_2 x [H^*(G_1; x) - p_1]$.  

**Case 4.** If $s \in V_2^{(i)}$ and $t \in V_2^{(j)}$ for $i, j = 1, 2, \ldots, p_1$, $i \neq j$, then we get the polynomial $p_2 x^2 [H^*(G_1; x) - p_1]$.  

Now, adding the polynomials obtained from the above cases and simplifying, we get the required result. ■
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