A Comparison for Some of the Estimators of Rayleigh Distribution with Simulation

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1. Abstract

In this paper , we use some statistical estimation methods to estimate the an unknown parameter of Rayleigh distribution to Know which is the best (the method that has less error).Bytaking maximum likelihood estimation , Bayes estimation , shrinkage estimation , and Bayesian shrinkage estimation . After then we applied the results in simulation study to compare these results , and obtain which method is the best .

الخلاصة :

في هذه الدراسة , استخدمنا بعض طرائق التقدير لتقدير معلمة توزيع Rayleigh لمعرفة الأفضل (الطريقة التي تعطي أقل خطأ ممكن) . إن الطرائق المستخدمة هي : طريقة تقدير الإمكان الأعظم والتقدير البيزي و البيزي المتقلص . لقد تم تطبيق النتائج في دراسة المحاكاة للمقارنة فيما بينها والحصول على أفضل طريقة .

2. Keywords

Maximum Likelihood Estimator , Bayes Estimator , Shrinkage Estimator , Bayesian Shrinkage Estimator , Rayleigh Distribution .

3. Notations

pdf	Probability Density Function
MLE	Maximum Likelihood Estimator
σ	Parameter
$\hat{\sigma}_{ml}$	Estimator of σ by ML
n	Sample Size
$\hat{\sigma}_{sh}$	Shrinkage Estimator of σ
$l(x;\sigma)$	Likelihood Function
k	Constant
Γ	Gamma Distribution
$\hat{\sigma}_{b}$	Bayesian Estimator of σ
$\hat{\sigma}_{bs}$	Bayesian Shrinkage Estimator ofσ

4. Introduction

The main branch of statistical inference is an estimation.There are some procedures of estimation . Some of these procedures which depend on a number of samples are called the classical methods like a maximum likelihood estimators , the other procedures depend on a prior information are called the Bayesian methods like Bayes estimators .

Some methods combine between these types are called shrinkage methods .

In this paper , we discuss the maximum likelihood estimation ,Bayesian estimation , shrinkage estimationand combine between them in Bayesian shrinkage estimation , and the comparison is done among them in a simulation study by using MATLAB program .
The Rayleigh distribution has a wide range of applications including lifetesting experiments and clinical studies.However, one parameter Rayleigh Distribution with probability density function (pdf) is given by:

$$f(x|\sigma) = \frac{x}{\sigma^2} e^{\left(\frac{-x^2}{2\sigma^2}\right)} \quad , x, \sigma > 0 \ldots (1)$$

5.Some Properties of RayleighDistribution [6]

- Mean

$$\mu(X) = \sigma \sqrt{\frac{\pi}{2}}$$

- Variance

$$Var(X) = \frac{4-\pi}{2}\sigma^2$$

- Mode

$$Mo = \sigma$$

- Moment generating function (MGF)

$$M(t) = 1 + \sigma t e^{-\frac{1}{2}\sigma^2 t^2} \sqrt{\frac{\pi}{2}} \left[erf\left(\frac{\sigma t}{\sqrt{2}}\right) + 1 \right]$$

Where,
erf : is the error function

6.Maximum Likelihood Estimation[5] [3]

The method of maximum likelihood (Harter and Moore (1965a), Harter and Moore (1965b), and Cohen (1965) is a commonly used procedure because it has very desirable properties .
Let x_1 , x_2 , x_3 , … , x_n be independent a random variables of size n , we assumed that the likelihood function (LF) of the probability density function of (1) is :

$$l(x;\sigma) = \prod_{i=1}^{n} \left[\frac{x_i}{\sigma^2} e^{\frac{-x_i^2}{2\sigma^2}} \right]$$

$$\ln l = \ln \left[\prod_{i=1}^{n} \left[\frac{x_i}{\sigma^2} e^{\frac{-x_i^2}{2\sigma^2}} \right] \right]$$

After simply we get :

$$\ln l = \sum_{i=1}^{n} \ln x_i - 2 \sum_{i=1}^{n} \quad ln\sigma - \frac{-\sum_{i=1}^{n} x_i^2}{2\sigma^2} \quad \ldots (2)$$

Now, we derivative equation (2) with respect to σ and equal to zero to get the maximum likelihood estimator for σ :

$$\hat{\sigma}_{ml} = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{2n}}$$

7.Shrinkage Estimation [3][6][2]

The shrinkage method is the link bridge between the classical estimators and the prior available information to the unknown parameter but in the form of initial values of σ_0.

The researcher's aim is to get the estimations of high-efficiency for theclassical estimators .The first most basic ideas in this direction were estimated by Goodman (1953) who suggested the following to estimate the parameter σ.

$$\tilde{\sigma} = k\sigma$$

After that , Thompson proposed the following estimate :

$$\tilde{\sigma} = k\hat{\sigma} + (1-k)\sigma_0 \quad \dots (3)$$

The equation of shrinkage is :

$$\hat{\sigma}_{sh} = k\hat{\sigma}_{ml} + (1-k)\sigma_0$$

Where ,
K is a constant between zero and one .
Then,the shrinkage estimator is :

$$\hat{\sigma}_{sh} = k\sqrt{\frac{\sum_{i=1}^{n} x_i^2}{2n}} + (1-k)\sigma_0$$

8.Bayesian Estimation[5][3][6] [4]

Let x_1 , x_2 , x_3 , ... , x_n, be a random sample of size n with distribution function F(x,σ) and the probability density function f(x, σ) , there are several steps to calculate the Bayes estimators of the Rayleigh distribution with one parameters σ, . So to do this must we must to know the prior distribution and posterior distribution as follows :

$$posterior\ distribution = \frac{prior\ \ distribution\ \ x\ \ likelihood\ function}{marginal\ distribution}$$

The conditional probability density function of σ given the data is given by :

$$\pi(\sigma|x) = \frac{2\left(\frac{s^2}{2}\right)^{n+c-\frac{1}{2}}}{\Gamma\left(n+c-\frac{1}{2}\right)(\sigma^2)^{n+c}} e^{\left[-\frac{s^2}{2\sigma^2}\right]}, \sigma>0$$

Where ,

$$s^2 = \sum_{i=1}^{n} x_i^2$$

c: constant

Note that σ^2 follows an inverted Gamma distribution , denoted as lnGa(α,β) which has the following from of the pdf :

$$f(u) = \frac{\beta^2}{\Gamma(\alpha)} u^{-(\alpha+1)} e^{\left[\frac{-\beta}{u}\right]} \; u, \alpha, \beta > 0$$

In our set up $\alpha = n + c - \frac{1}{2}$, and $\beta = \frac{s^2}{2}$

By usingsquared errorloss function given by :

$L_1(\hat{\sigma}, \sigma) = (\hat{\sigma}; \sigma)^2$,the risk function is :

$$R_s(\hat{\sigma}_b) = \int_0^{\infty} L_1(\hat{\sigma}, \sigma)\pi(\sigma|x)d\sigma$$

$$= \hat{\sigma}_b^2 - 2\hat{\sigma}_b \frac{\Gamma(n+c-1)}{\Gamma\left(n+c-\frac{1}{2}\right)} \sqrt{\frac{s^2}{2}} + \frac{\Gamma\left(n+c-\frac{3}{2}\right)s^2}{\Gamma\left(n+c-\frac{1}{2}\right)^2}$$

The Baye's estimator $\hat{\sigma}_b$ is a solution of the equation ,$\frac{\partial R(\hat{\sigma}\;\;)}{\partial \hat{\sigma}}$, which implies ,

$$\hat{\sigma}_b = \frac{\Gamma(n+c-1)}{\Gamma\left(n+c-\frac{1}{2}\right)} \sqrt{\frac{s^2}{2}}$$

Where,

$$s^2 = \sum_{i=1}^{n} x_i^2$$

c: constant

9.Bayesian Shrinkage Estimation[3][2]

The Bayesian shrinkage estimator combines them (Bayesian andshrinkage method).

So , we will use the Bayesian estimator as a prior information .

Now , to combine the Bayes and shrinkage estimators we get :

$\hat{\sigma}_{bs} = k\hat{\sigma}_b + (1 - k)\sigma_{ml}$

Then, the Bayesian shrinkage estimator is :

$$\hat{\sigma}_{bs} = k \frac{\Gamma(n + c - 1)}{\Gamma\left(n + c - \frac{1}{2}\right)} \sqrt{\frac{s^2}{2}} + (1 - k) \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{2n}}$$

10.Simulation Study[1]

There are many methods of simulation (especially after the rapid development that took place in the use of electronic computers), which provides the time, effort ,cost and achieve analytical solutions . Simulation is the imitation of the operation of a real-world process or system over time. The act of simulating something first requires a model to be developed; this model represents the key characteristics , behaviors of the selected physical , abstract system ,or process. The model represents the system itself, while the simulation represents the operation of the system over time . Computer simulations have become a useful part of mathematical modeling of many natural systems in sciences .So , the simulation is a type of sampling techniques . The process of simulation strategy is explained as in Figure (1), and the numerical results in the Table(1) , Table(2) and Table(3) .

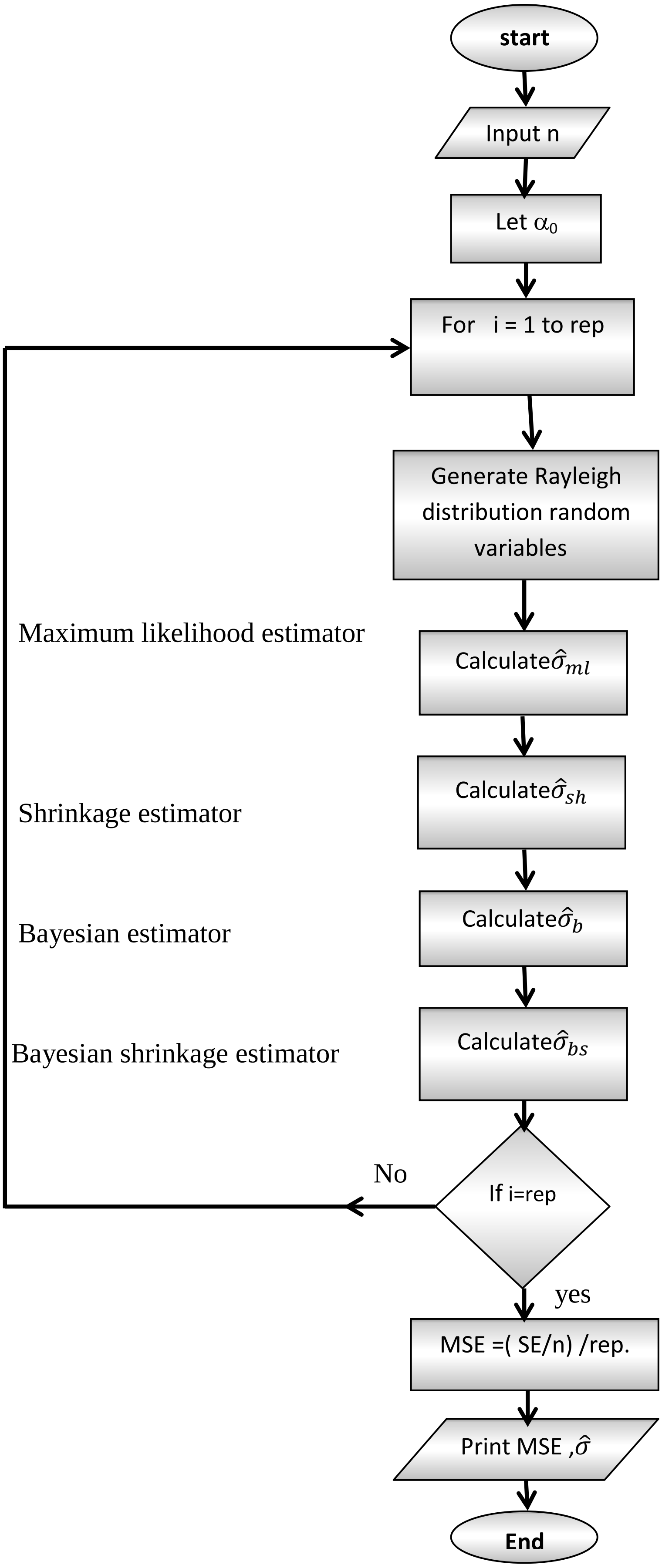

Fig (1) : Flow chart for The strategyof Simulation procedure

Table (1) : Results of Simulation using σ_0=1 , k=0.3 :

Sample size	$\hat{\sigma}_{ml}$	MSE_{ml}	$\hat{\sigma}_{sh}$	MSE_{sh}	$\hat{\sigma}_b$	MSE_b	$\hat{\sigma}_{bs}$	MSE_{bs}
10	1.1344	0.0056	1.0403	0.0063	1.0940	0.0059	1.0282	0.0064
30	0.8880	0.0103	0.9664	0.0093	0.8771	0.0105	0.9631	0.0094
60	0.9885	0.0305	0.9966	0.0303	0.9824	0.0307	0.9947	0.0303
100	1.0409	0.0579	0.0579	0.0593	1.0370	0.0581	1.0111	0.0594

Table (2) : Results of Simulation using σ_0=1 , k=0.5 :

Sample size	$\hat{\sigma}_{ml}$	MSE_{ml}	$\hat{\sigma}_{sh}$	MSE_{sh}	$\hat{\sigma}_b$	MSE_b	$\hat{\sigma}_{bs}$	MSE_{bs}
10	1.1344	0.0056	1.0672	0.0061	1.0940	0.0059	1.0470	0.0062
30	0.8880	0.0103	0.9440	0.0096	0.8771	0.0105	0.9385	0.0096
60	0.9885	0.0305	0.9943	0.0304	0.9824	0.0307	0.9912	0.0304
100	1.0409	0.0579	1.0204	0.0589	1.0370	0.0581	1.0185	0.0590

Table (3) : Results of Simulation using σ_0=1 , k=0.7 :

Sample size	$\hat{\sigma}_{ml}$	MSE_{ml}	$\hat{\sigma}_{sh}$	MSE_{sh}	$\hat{\sigma}_b$	MSE_b	$\hat{\sigma}_{bs}$	MSE_{bs}
10	1.1344	0.0056	1.0941	0.0059	1.0940	0.0059	1.0658	0.0061
30	0.8880	0.0103	0.9216	0.0098	0.8771	0.0105	0.9140	0.0099
60	0.9885	0.0305	0.9920	0.0304	0.9824	0.0307	0.9877	0.0305
100	1.0409	0.0579	1.0286	0.0585	1.0370	0.0581	1.0259	0.0586

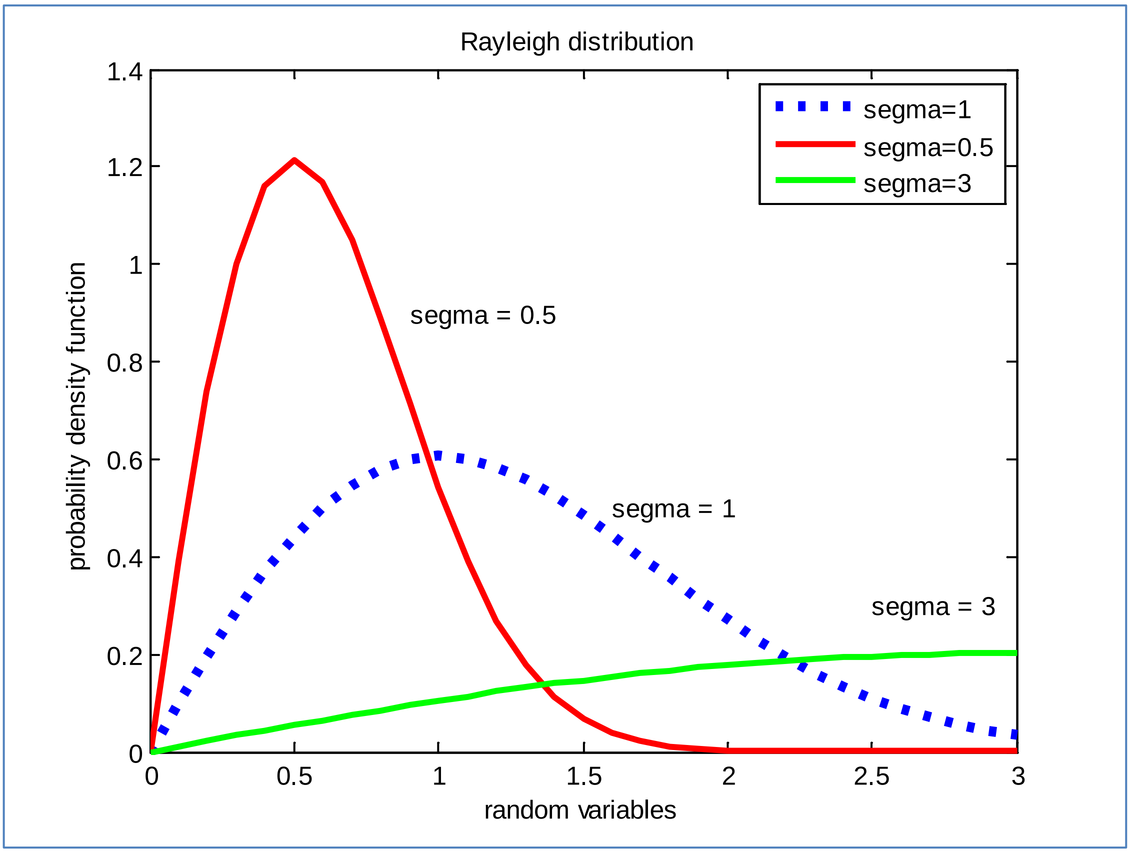

Fig (2) : Probability Density Function of Rayleigh Distribution with different values of σ

The abovegraph in Figure (2) is referring to plot of probability density function of Rayleigh distribution for data (the random variables are generated in MATLAB), we see different values of σ and then , different curves of Probability density function .The blue curve represent the pdf when (σ=1), the red curve when (σ=0.5),and the green one when (σ=3).

11.Conclusions

In this paper ,we have used some methods of estimation are maximum likelihood , Bayesian , shrinkage and Bayesian shrinkage estimator . In simulation study , we compare the results and we fined :

1. The mean squares error (MSE) Variant to one sample to another .
2. Not always , The Bayesian estimator the best of ML .
3. In this study , MLE the best estimated of Bayesian in all sizes of samples and that by reason to the random samples or prior information .
4. When k increasing , the Bayesian shrinkage estimator become best .
5. In all sample sizes , the shrinkage estimator the shrinkage estimator is the best of the Bayesian shrinkage estimator .

12.References

1. Davis Timothy & A. ,Sigmon K.,"MATLAB Primer",CHAPMAN&HALL/CRC,2005.
2. Dey S. ," Bayes Shrinkage Estimation of the Parameter of Rayleigh Distribution for Progressive Type-II Censored Data" ,Banaras Hindu University Varanasi, Uttar Pradesh, INDIA **,** 2012 .
3. Dey S. "Bayesian Estimation of the Parameter and ReliabilityFunction of an Inverse Rayleigh Distribution",Malaysian Journal of Mathematical Sciences 6(1):113-124 ,2012 **.**
4. Ethem Al. " Introduction to Machine Learning: , 2ed Edition, The MIT Press Cambridge , Massachusetts London , England,2010 .
5. Larsen R.J. ,and Marx M.L. ,"Mathematical statistics and applications", PERSON , 5th Edition , 2012.
6. Sinha S.K., and Kale B.K. "LifeTesting and Reliability Estimation",WILEYEASTERN LIMITED , 1980 .