A COMPARATIVE STUDY OF BOOTSTRAP AR(1) PARAMETER ESTIMATIONS

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ABSTRACT

Its well-known that the target of all statisticians is to get accurate statistical inference. Bootstrap methods are more widely applicable than other re-sampling procedures to measure the statistical accuracy for parameter estimators. Efron in 1979 was introduce bootstrap for estimating sampling distributions based on a finite sample of i.i.d. observations[(4)]. The assumption of i.i.d is violated when the observations are serially correlated. To overcome this problem, several solutions are introduced. In this paper, we compare between MLE and the bootstrap procedures in case of strictly stationary autoregressive model. Theil U statistic is used as a criterion to make the comparison. Some real data are studied furthermore the simulation.

Keywords: AR(p), MLE, Bootstrap, Bootstrap residuals, Theil U statistic.

1. Introduction

In 1979, Efron introduced the bootstrap as a computer-intensive statistical method, where the finite sample of observations are identically independent distributed (i.i.d.)[(5)]. Sometimes, the sampling distribution is difficult to derive or even asymptotically therefore, the alternative is the non-parametric methods that are based on free distributions. The classical bootstrap is one of non-parametric statistical tools that is assumed replacing the population distribution by empirical one as a consistent estimation. So, the bootstrap can describe the variability of a statistic without distributional assumptions about data. For example, approximation Maximum Likelihood Estimation (MLE) that used to estimate the *parameters of time series model requires the process is normally distributed. However, in real world many processes are not normal, therefore one of appropriate procedures to approximate results is the bootstrap*[11]*. But there is another problem occur when we want to apply bootstrap in time series analysis, that the observations are restricted dependent or serially correlated. In this case the assumption of (i.i.d.) observations is violated.*

1.1Autoregressive Model Order of (P). AR(P)

The general form of Autoregressive model AR(P) can be written as

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + \varepsilon_t \quad (1)$$

where $\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$.

and $\phi_j \quad j = 1,2..., p$ *represent the model parameters, if* $p = 1$ *the model is AR(1) or is called Markov model.*

$$Y_t = \phi_1 . Y_{t-1} + \varepsilon_t, \quad t = 1,2,..., n \quad (2)$$

The process will be stationary if $|\phi_1| < 1$*, when* $|\phi_1| > 1$ *its non-stationary, while* $|\phi_1| = 1$ *the process is called random walk.*

1.2 Approximate Maximum Likelihood (MLE)

From equation (2)

$$\begin{aligned} \varepsilon_2 &= Y_2 - \phi_1 Y_1 \\ \varepsilon_3 &= Y_3 - \phi_1 Y_2 \\ &\vdots \\ \varepsilon_t &= Y_t - \phi_1 Y_{t-1} \end{aligned} \quad (٣)$$

Since ε_t*'s are independent then the MLE can be used to find that*

$$f(\varepsilon_t) = (2\pi\sigma_\varepsilon^2)^{-0.5} EXP\left(-\frac{1}{2\sigma_\varepsilon^2} \varepsilon_t^2 \right) \quad (٤)$$

and the likelihood,

$$L=\left(2\pi\sigma_{\varepsilon}^{2}\right)^{-\frac{n}{2}}EXP\left(-\frac{1}{2\sigma_{\varepsilon}^{2}}\sum_{t=2}^{n}\varepsilon_{t}^{2}\right)$$

(٥)

by using the transformation we find from (3)

$$|J|=\left|\frac{\partial\varepsilon_{t}'s}{\partial Y_{t}'s}\right|=1$$

(٦)

thus

$$L=\left(2\pi\sigma_{\varepsilon}^{2}\right)^{-\frac{n}{2}}EXP\left(-\frac{1}{2\sigma_{\varepsilon}^{2}}\sum_{t=2}^{n}\left(Y_{t}-\phi_{1}Y_{t-1}\right)^{2}\right)$$

(٧)

$$\frac{\partial\ln L}{\partial\phi_{1}}=\frac{1}{\sigma_{\varepsilon}^{2}}\sum_{t=2}^{n}\left(Y_{t}-\phi_{1}Y_{t-1}\right)Y_{t-1}$$

(٨)

by $\frac{\partial\ln L}{\partial\phi_{1}}=0$

then,

$$\sum_{t=2}^{n}\left(Y_{t}Y_{t-1}-\hat{\phi}_{1}Y_{t-1}^{2}\right)=0$$

$$\therefore\hat{\phi}_{1}=\frac{\sum_{t=2}^{n}Y_{t}Y_{t-1}}{\sum_{t=2}^{n}Y_{t-1}^{2}}$$

(٩)

1.3 The Bootstrap Approach to Parameter Estimation

The bootstrap is a form of a larger class of methods that resample from the original dataset with replacement. It accepts to be more powerful tool for statistical inference. Efron and Tibshirani (1993) introduced a text book, which was the first attempt to present the general methodology and applications of bootstrap to multidisciplinaries. Bootstrap re-samples

obtained by independently sampling with replacement from the empirical distribution that is the simple estimate of the entire distribution. From the *bootstrap sampling, a Monte Carlo approximation of the bootstrap estimate can be obtained as follows,*

- *Generate a sample with replacement from the empirical distribution (a bootstrap sample),*
- *Compute * the value of $\hat{\theta}$ obtained by using the bootstrap sample in place of the original sample,*
- *Repeat the two steps above k times.*

In regression setting there are two basic approaches of conducting bootstrap procedure; the fixed-x re-sampling (Bootstrap 1) and the random-x re-sampling (Bootstrap 2). Fixed-x re-sampling is to first fit the model and bootstrap the residuals. Random-x re-sampling is to bootstrap the vector of the response variables and the associate predictor variables (see: Imon and Ali; 2005, Cherinck; 2008)[3,7].

The bootstrap methods mentioned above are only appropriate the residuals are not serially correlated. Since the. assumption of i.i.d observations is violated in most econometric applications with time series data[1]. Several approaches have proposed to apply bootstrap procedure with serially correlated data in stationary case .

The first approach, Bootstrap 1 is to an approximate i.i.d. setting by focusing on the residuals which resampled instated of original observations. Although, fitted residuals is faster than i.i.d. observation in general, but exhibits some form of heteroskedasticity. Freedman 1981[6];Liu 1988[10] showed that the classical bootstrap perform reasonably well even when the data are independent but not identically distributed therefore as well, perhaps certain the robust degree to heteroskedasticity[8].

The second approaches Bootstrap 2 is based on blocking techniques in which the data are divided in blocks of observations with appropriate length and these blocks are resampled. In this situation the dependency in consecutive observations will be reducing or weakly. Its well known, there are three block bootstrap methods: moving block bootstrap (MBB) method,

non-overlapping block bootstrap (NBB) method and circular block bootstrap method (CBB). For more information read: Carlstein (1986), Kunsch (1989) and Liu and Singh (1992)[2,12,10].

Efron and Tibshriani (1993) used Bootstrap 1 procedure to resample directly from the estimated residuals. This approach is justified when the mean of $\hat{F}(.)$ is close to zero. If it is not, Efron and Tibshriani (1993) suggested that the definition $\mathrm{E}_F(\varepsilon_t)=0$ could be honored by centering the residuals around their mean. Freedman(1981) concluded that without centering, the bootstrap will usually fail. Souza and Neto(1996) showed in their simulation study that the degrees of freedom lost in the fitting process. So, they inflate the estimated residuals by the factor $[n/(n-k)]^{1/2}$, where k is the number of free parameters of the model. Shao (1996) combined these two ideas by generating i.i.d. $\varepsilon_1^*,...,\varepsilon_n^*$ from free distribution putting mass n^{-1} on $(\hat{\varepsilon}_t-\bar{\varepsilon}_t)/\sqrt{1-p/n}$ where the $\bar{\varepsilon}$ is the average of the $\hat{\varepsilon}_t$ and P is the order of the model.[8]

2. MATERIALS AND METHODS

2.1 Residual Bootstrap in AR(1) Model:

Let Y_t be a stationary AR(1) process, that is

$$Y_t=\phi_1.Y_{t-1}+\varepsilon_t, \quad t=1,2,...,n$$

Assume that σ to be known, and ϕ_1 is the unknown parameter of interest. Then the Least Squares estimate for ϕ_1 (which is approximately the MLE in case of normal errors) is given by

$$\hat{\phi}_1 = \frac{\sum_{t=2}^{n} Y_t Y_{t-1}}{\sum_{t=2}^{n} Y_{t-1}^2}$$

Let $\tilde{\varepsilon}_t=Y_t-\hat{\phi}_1 Y_{t-1}, \quad t=1,2,...,n$ and $\hat{\varepsilon}_t=(\tilde{\varepsilon}_t-\bar{\varepsilon}_t)/\sqrt{1-p/n}$ where $\bar{\varepsilon}_t=\frac{\sum_{t=1}^{n}\tilde{\varepsilon}_t}{n}$, then bootstrap procedure residuals as follow, 1- *Draw* $\varepsilon_t^*, t = 1,2,...n$ *random sample with replacement from* $\hat{\varepsilon}_t$ *and define*

$$Y_1^* = \varepsilon_1^*$$
$$Y_t^* = \hat{\phi}_1 \, Y_{t-1}^* + \varepsilon_t^*$$

2- *For a bootstrap sample find the statistic,*

$$\hat{\phi}_{1b}^* = \frac{\sum_{t=2}^{n} Y_t^* Y_{t-1}^*}{\sum_{t=2}^{n} (Y_{t-1}^*)^2} \qquad b = 1,\ldots, B$$

(١٠)

3- *Repeat 1 and 2 steps B times.*

4- *The bootstrap the variance , bias and MSE can be estimated as follows;*

$$Var_{boot} = \frac{1}{B-1} \sum_{b=1}^{B} (\hat{\phi}_{1b}^* - \hat{\phi}_{boot})^2$$

where

$$\hat{\phi}_{boot} = \frac{1}{B} \left(\sum_{b=1}^{B} (\hat{\phi}_{1b}^*) \right)$$

$$Bias = \left(\hat{\phi}_{boot} - \phi_1 \right)$$

$$MSE = Bias^2 + Var_{boot}$$

(١١)

2.2 <u>Theil's U-Statistic</u>

Theil suggested this criterion in 1966 to merge the properties of percentage criterions as Mean Percentage Errors (MPE) , Mean Absolute Percentage Errors (MAPE) criterions ,and their square values to get positive value. In this situation the comparison among the different methods will be easy. The mathematical form of this statistic can give as follow,

$$U = \sqrt{\frac{\sum_{t=2}^{n} \left(\frac{\hat{Y}_{t+1} - Y_t}{Y_t} \right)^2}{\sum_{t=2}^{n} \left(\frac{Y_{t+1} - Y_t}{Y_t} \right)^2}}$$

(١٢)

3. Result and Discussion

3.1 Simulations

The aim of bootstrap 1 is to identify an AR(1) model to accurate the forecasting of observations with unknown standard error. The performance of bootstrap is judged by comparing bootstrap results of MSE and Theil statistic to original one.

Let

$$\phi = \begin{Bmatrix} 0.1 \\ 0.5 \\ 0.9 \end{Bmatrix} \qquad \varepsilon_t \sim \begin{Bmatrix} N(0,1) \\ t(1), t(3) \end{Bmatrix}$$

Table (1) :Residuals Bootstrap with AR(1) and the residual is distributed $\varepsilon_t \overset{iid}{\sim} N(0,1)$

N=	**15**			**50**			**100**			**500**		
ϕ_1	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9
$\hat{\phi}_1$	-0.12	0.302	0.865	0.026	0.476	0.916	0.098	0.499	0.900	0.011	0.520	0.907
$\hat{\phi}_{boot}$	-0.08	0.335	0.802	0.036	0.469	0.889	0.109	0.509	0.904	0.011	0.523	0.907
$MSE(\hat{\phi}_1)$	0.119	0.104	0.019	0.026	0.016	0.004	0.010	0.008	0.002	0.002	0.002	0.000
$MSE(\hat{\phi}_{boot})$	0.098	0.088	0.050	0.020	0.013	0.005	0.008	0.004	0.001	0.002	0.002	0.000
$Theil(\hat{\phi}_1)$	0.326	0.028	0.475	0.021	0.037	0.037	0.025	0.015	0.006	0.041	0.014	0.008
$Theil(\hat{\phi}_{boot})$	0.314	0.026	0.695	0.021	0.038	0.050	0.025	0.014	0.005	0.041	0.014	0.008

Table (2) :Residuals Bootstrap with AR(1) and the residual is distributed $\varepsilon_t \overset{iid}{\sim} t(1)$

N	**15**			**50**			**100**			**500**		
ϕ_1	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9
$\hat{\phi}_1$	0.044	0.418	0.879	0.254	0.583	0.926	0.133	0.485	0.859	0.109	0.532	0.916
$\hat{\phi}_{boot}$	0.1	0.4	0.8	0.2	0.5	0.9	0.1	0.4	0.8	0.1	0.5	0.9

	40	84	83	66	79	01	46	95	66	10	34	15
$MSE(\hat{\phi}_1)$	0.075	0.066	0.017	0.043	0.020	0.004	0.011	0.008	0.004	0.002	0.002	0.001
$MSE(\hat{\phi}_{boot})$	0.052	0.040	0.024	0.045	0.019	0.005	0.010	0.007	0.004	0.001	0.002	0.001
$Theil(\hat{\phi}_1)$	0.068	0.093	0.029	0.016	0.057	0.017	0.019	0.032	0.053	0.024	0.005	0.019
$Theil(\hat{\phi}_{boot})$	0.061	0.083	0.028	0.015	0.059	0.023	0.019	0.031	0.050	0.024	0.005	0.019

Tabe(3):Residuals Bootstrap with AR(1) and the residual is distributed $\varepsilon_t \overset{iid}{\sim} t(3)$

N	15			50			100			500		
ϕ_1	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9
$\hat{\phi}_1$	-0.176	0.162	0.672	0.107	0.475	0.921	0.064	0.429	0.803	0.110	0.533	0.920
$\hat{\phi}_{boot}$	-0.102	0.218	0.634	0.156	0.533	0.931	0.079	0.437	0.803	0.110	0.531	0.920
$MSE(\hat{\phi}_1)$	0.145	0.184	0.091	0.020	0.017	0.004	0.011	0.013	0.013	0.002	0.003	0.001
$MSE(\hat{\phi}_{boot})$	0.095	0.121	0.051	0.018	0.011	0.003	0.010	0.011	0.013	0.002	0.002	0.001
$Theil(\hat{\phi}_1)$	0.005	0.183	0.108	0.018	0.078	0.049	0.020	0.036	0.069	0.024	0.006	0.012
$Theil(\hat{\phi}_{boot})$	0.005	0.171	0.088	0.018	0.069	0.041	0.020	0.036	0.069	0.024	0.006	0.012

3.2 Real Data:

3.2.1 Al-Diwanyia Textile Factory Sales Data (2005-2010):

Table (4) shows the sales of Al-Diwanyia Textile factory by Iraqi dinnar for the period (2005-2010). We begin with the first 24 months (2005-2006) then the length of time series increases 24 months step by step to arrive 72 months for (2005-2010). Actually, we attempted to show the efficiency of bootstrap technique for the short or tall length of time series.

Table (4): Results of Al- Diwanyia Textile Factory Sales Data

The Time Length(Monthly)	***24 (2005-2006) ID***	***48 (2005-2008) ID***	***72 (2005-2010) ID***
$\hat{\phi}_1$	***0.761***	***0.729***	***0.899***
$\hat{\phi}_{boot}$	***0.702***	***0.692***	***0.878***
$MSE(\hat{\phi}_1)$	***0.086***	***0.063***	***0.162***
$MSE(\hat{\phi}_{boot})$	***0.065***	***0.049***	***0.147***
$Theil(\hat{\phi}_1)$	***0.019***	***0.017***	***0.009***
$Theil(\hat{\phi}_{boot})$	***0.019***	***0.017***	***0.009***

3.2.2 Al-Diwanyia Textile Factory Production Quantity Data (2005-2010):

Table (5) the production quantity of Al-Diwanyia Textile factory by matter scale for the period (2005-2010). At the same procedure we divided the time series for three parts in which the first part for (2005-2006) monthly then increase the length 24 months step by step to the last month of 2010.

.Table (5): Results of Al- Diwanyia Textile Factory Production Quantity

The Time Length (Monthly)	***24 (2005-2006)***	***48 (2005-2008)***	***72 (2005-2010)***
$\hat{\phi}_1$	***0.390***	***0.391***	***0.847***
$\hat{\phi}_{boot}$	***0.395***	***0.395***	***0.826***
$MSE(\hat{\phi}_1)$	***0.049***	***0.026***	***0.124***
$MSE(\hat{\phi}_{boot})$	***0.044***	***0.022***	***0.111***
$Theil(\hat{\phi}_1)$	***0.200***	***0.018***	***0.009***

$Theil(\hat{\phi}_{boot})$	**0.200**	**0.018**	**0.009**

4. Conclusion

4.1 Simulation

From tables (1-3) we can conclude as follow:

1- $\varepsilon_t \overset{iid}{\sim} N(0,1)$

There was a close results when we use MLE and bootstrap especially where $\phi_1 = 0.1, 0.5$ *, this closeness increases when the sample size increases ,thus MSE and Theil values tend to be very close for two methods at n=500.*

2- $\varepsilon_t \overset{iid}{\sim} t(1), \varepsilon_t \overset{iid}{\sim} t(3)$

In this case the assumption of error normality distribution was violated , which caused increasing in MSE and Theil values for the two methods in comparison with previous case when $\varepsilon_t \sim N(0,1)$*. Furthermore the increasing was larger when* $\phi_1 \to 0.1$ *,also we can find that in small samples the bootstrap gives better results than the MLE method.*

3- Generally we conclude from tables (1-3) that MSE and Theil decreasing when sample size increasing, and their values tend to be very close for MLE and bootstrap, so this case coincide with the statistical theory.

4.2 Real Data

We can conclude from the tables (4) and (5) that the bootstrap method had given better results than the method of MLE, this can be seen from the values of MSE. We denote also that these values are close to the values of

simulation results. Thiel statistics results are approximated that refers to accurate best estimators for

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