بعض مبرهنات النقطة الصامدة لتطبيق ذاتي منكمش معمم في فضاء كون المتري

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المستخلص
في هذا البحث، تم برهان بعض مبرهنات النقطة الصامدة لتطبيق ذاتي يحقق شرطاً منكمشاً معمماً في فضاء كون المتري مع افتراض ان الكون المستخدمة في هذا الفضاء غير طبيعية. نتائجنا هي تعميم لبعض النتائج الحالية.

Some Fixed Point Theorems for Generalized Contractive Self Mapping on Cone Metric Space

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Abstract

In this paper, we prove some fixed point theorems for self mapping satisfying generalized contractive condition in the setting of Cone metric space with assumption that the Cone is non-normal. Our results are generalizations of some recent results.

Key Words: Fixed point, generalized contractive self mapping, Cone metric space.

1 - Introduction

Haung and Zhang in [1] was extended the concept of metric space and introduced the concept of Cone metric space by replacing the set of real numbers by an ordered Banach space by using the normality of a Cone and obtained some fixed point results, Rezapour and Hamlbarani in [2] generalized some results of [1] by omitting the assumption of normality of a Cone in the results which is a milestone in developing fixed point theory in Cone metric space. Recently, the author in [3] was generalized the results in [2] by establishing some new generalized contractive type conditions for self mappings defined on Cone metric spaces and prove some new fixed point theorems for these mappings by using certain function of the family of functions denoted by $\tilde{U}(p,p)$ satisfying some properties where the normality of the Cone is omitted in these results.

Now in this paper we generalize the results in [3] by establishing a generalized contractive type condition for self mapping defined on non-normal Cone metric space, this type of contractive condition is more extended of the generalized contractive type conditions which was used in the results of [3], such that we used in this type of extended contraction the function of the family $\tilde{U}(p,p)$ which was appeared in [3]. In the same time, our results are generalizations of some other recent results.

2 – Preliminaries

<u>Definition 2.1</u>: [1]

Let E be a real normed space and P be a subset of E, P is called a Cone if:

(a) P is closed, non empty and $P \neq \{0\}$.
(b) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b.
(c) $P \cap (-P) = \{0\}$.

Given a Cone $P \subseteq E$, we define a partial ordering "$\leq$" with respect to P by $x \leq y$ if and only if $y - x \in P$, we shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$ and for $x, y \in P$, $x \ll y$ stand for $y - x \in int(P)$, where int(P) is the interior of P.

The Cone P is called normal (with respect to this norm of E) if there exists a number $k > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq k\|y\|$, the least positive number satisfying the above inequality is called the normal constant of P.

<u>Example (2.2):</u> [2]

Let $E = C_R([0,1])$ with supremum norm and $P = \{f \in E: f \geq 0\}$, where

$\|f\| = \sup\{|f(x_i)|, x_i \in [0,1]\}$ for all f, g $\in$ P, put f(x) = x, g(x) = 2x, then $0 \le f \le g$, $\|f\| = 1$, $\|g\| = 2$. So $\|f\| \le \|g\|$ and k = 1. Therefore P is normal Cone with normal constant k = 1.

Remark (2.3): [2]

There are Cones are not normal, the following example show that:

Example (2.4): [2]

Let $E = C^2_R([0,1])$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$ and consider the Cone $P = \{f \in E: f \ge 0\}$, where

$\|f\|_\infty = \max\{|f(x_1)|, | f(x_2)|, \ldots, |f(x_n)|, x_i \in [0,1] \ \forall\ i = 1,2,\ldots,n\}$

$\|f\|_\infty = \max\{|f(x_1)| + | f(x_2)| + \ldots + |f(x_n)|, x_i \in [0,1] \ \forall\ i = 1,2,\ldots,n\}$

For each $k \ge 1$, put f(x) = x and $g(x) = x^{2k}$. Then $0 \le g \le f$, $\|f\| = 2$ and $\|g\| = 2k + 1$, since $k\|f\| < \|g\|$, k is not a normal constant of P. Therefore, P is non-normal Cone.

Definition (2.5): [1]

Let X be non-empty set and (E,P) is an real normed space), a mapping $d: X \times X \longrightarrow E$ is called a Cone metric on X if the following conditions are satisfied:

(i) $0 \le d(x,y)$ for all x, y $\in$ X and d(x,y) = 0 if and only if x = y.

(ii) d(x,y) = d(y,x) for all x, y $\in$ X.

(iii) $d(x,y) \le d(x,z) + d(z,y)$ for all x, y, z $\in$ X.

Then the ordered pair (X,d) is called Cone metric space.

Example (2.6): [1]

Let $E = R^2$ with usual norm on R^2 defined by $\|x\| = \max\{|x_1|, |x_2|\}$ for all $x \in R^2$, $x = (x_1,x_2)$, $x_i \in R$, i = 1, 2, $P = \{(x,y) \in E: x, y \ge 0\} \subset R^2$, X = R and $d: X \times X \longrightarrow E$ such that: $d(x,y) = (|x - y|, \alpha|x - y|)$, where $\alpha \ge 0$ is a constant. Then (X,d) is a Cone metric space.

Solution:

(i) Since $|x - y| \ge 0$ for all $x, y \ge 0$ and $\alpha \ge 0$ therefore we have $0 \le d(x,y)$.

Now, if $d(x,y) = 0 \Leftrightarrow (|x - y|, \alpha|x - y|) = 0$

$\Leftrightarrow |x - y| = 0$

$\Leftrightarrow x - y = 0$

$\Leftrightarrow x = y$

(ii) $d(x,y) = (|x - y|, \alpha|x - y|) = (|-(y-x)|, \alpha|-(y-x)|) = (|y-x|, \alpha|y-x|) = d(y,x)$

(iii) $d(x,y) = (|x - y|, \alpha|x - y|)$

$= (|x - z + z - y|, \alpha|x - z + z - y|)$

$\leq (|x - z| + |z - y|, \alpha|x - z| + |z - y|)$ by properties of absolute value

$\leq (|x - z|, \alpha|x - z|) + (|z - y|, \alpha|z - y|)$

$\leq d(x,z) + d(z,y) = d(x,z) + d(y,z)$

Therefore (X,d) is Cone metric space.

For more examples of Cone metric spaces we refere [1,2].

Definition (2.7): [1]

Let (X,d) be a Cone metric space, let $\langle x_n \rangle$ be a sequence in X and $x \in X$, then

(1) $\langle x_n \rangle$ converge to x (denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$) if for any $c \in int(P)$, there exists N such that for all $n > N$, $d(x_n,x) \ll c$.

(2) $\langle x_n \rangle$ is Cauchy sequence if for any $c \in int(P)$, there exists N such that for all $n, m > N$, $d(x_n,x_m) \ll c$.

(3) (X,d) is complete if every Cauchy sequence in X is convergent.

In the following we always suppose that E is a real normed space, P is non-normal Cone in E with $int(P) \neq \phi$ and "≤" is a partial ordering with respect to P.

Definition (2.8): [4]

If Y be any partially ordered set with relation "≤" and $f : Y \longrightarrow Y$, we say that f is non-deceasing if, $x, y \in Y$, $x \leq y \Rightarrow f(x) \leq f(y)$.

Definition (2.9): [3]

A function $F:P \longrightarrow P$ is called ≪-increasing if for each $x, y \in P$, $x \ll y$ if and only if $F(x) \ll F(y)$.

Definition (2.10): [3]

A function $f: P \longrightarrow P$ is called subadditive if for all $x, y \in P$, $f(x + y) \leq f(x) + f(y)$.

Definition (2.11): [3]

Let $F: P \longrightarrow P$ be a function such that

(F_1) $F(t) = 0$ if and only if $t = 0$.

(F_2) F is $\ll$-increasing.

(F_3) F is surjective.

We denote by $\tilde{U}(p,p)$ the family of functions satisfying (F_1), (F_2) and (F_3).

Example (2.12):

Let $F(t) = t$, for each $t \in P$. Then $F \in \tilde{U}(p,p)$.

For more examples of $\tilde{U}(p,p)$ you can see [3].

Remark (2.13):

Since every normed space is a topological vector space, (i.e.) (normed space is a special case of topological vector space). So we can use the following lemma which is necessary through our work in this sequel, this lemma was appeared in [5].

Lemma (2.14): [5]

Let E be a topological vector space. If $c_n \in E$ and $c_n \longrightarrow 0$, then for each $c \in int(P)$ there exists N such that $c_n \ll c$ for all $n > N$.

proof: see [5]

3- Main Result

Theorem (3.1):

Let (X,d) be a complete Cone metric space, suppose that a mapping $T: X \longrightarrow X$ satisfyies:

$$F[d(Tx,Ty] \leq a_1F[d(x,y)] + a_2F[d(Tx,x)] + a_3F[d(Ty,y)] + a_4F[d(Tx,y)] + a_5F[d(Ty,x)] \quad \ldots(3.1.1)$$

For all $x, y \in X$, where a_i, $i = 1,2,3,4,5$ are all non-negative constants such that $\sum_{i=1}^{5} a_i < 1$ and $F \in \tilde{U}(p,p)$ such that

(1) F is non-decreasing and subadditive.

(2) If, for $<c_n> \subset P$, $\lim_{n\to\infty} F(c_n) = 0$ then $\lim_{n\to\infty} c_n = 0$.

Then T has a unique fixed point in X, for each $x \in X$, the iterative sequence $<T^n x>$ converges to the fixed point.

Proof:

Let $x_0 \in X$ be arbitrary. Let $x_1 = Tx_0$, and $x_{n+1} = Tx_n = T^{n+1}x_0$, for all $n \in N$

First we will show that the sequence $<x_n>$ is a Cauchy sequence.

Taking $x = x_n$, $y = x_{n-1}$ in equation 3.1.1; we get

$F[d(Tx_n,Tx_{n-1})] \le a_1F[d(x_n,x_{n-1})] + a_2F[d(Tx_n,x_n)] + a_3F[d(Tx_{n-1},x_{n-1})]+$

$a_4F[d(Tx_n,x_{n-1})] + a_5F[d(Tx_{n-1},x_n)]$

$F[d(x_{n+1},x_n)] \le a_1F[d(x_n,x_{n-1})] + a_2F[d(x_{n+1},x_n)] + a_3F[d(x_n,x_{n-1})]+ a_4F[d(x_{n+1},x_{n-1})] +$

$a_5F[d(x_n,x_n)]$

$F[d(x_{n+1},x_n)] \le a_1F[d(x_n,x_{n-1})] + a_2F[d(x_{n+1},x_n)] + a_3F[d(x_n,x_{n-1})]+ a_4F[d(x_{n+1},x_n)] +$

$a_4F[d(x_n,x_{n-1})]$

So, we get

$F[d(x_{n+1},x_n)] \le (a_1 + a_3 + a_4)\, F[d(x_n,x_{n-1})] + (a_2 + a_4)\, F[d(x_{n+1},x_n)]$ …(3.1.2)

Now, taking $y = x_n$, $x = x_{n-1}$ in equation (3.1.1) and using symmetry of inequality in x, y, we get:

$F[d(x_n,x_{n+1})] \le (a_1 + a_2 + a_5)F[d(x_{n-1},x_n)] + (a_3 + a_5)\, F[d(x_n,x_{n+1})]$ …(3.1.3)

Combining equations (3.1.2) and (3.1.3), we get:

$2\, F[d(x_{n+1},x_n)] \le (2a_1 + a_2 + a_3 + a_4 + a_5)\, F[d(x_n,x_{n-1})] +$

$(a_2 + a_3 + a_4 + a_5)\, F[d(x_{n+1},x_n)]$

$$F[d(x_{n+1},x_n)] \le \frac{2a_1 + a_2 + a_3 + a_4 + a_5}{2 - a_2 - a_3 - a_4 - a_5}\, F[d(x_n,x_{n-1})]$$

$F[d(x_{n+1},x_n)] \le \lambda\, F[d(x_n,x_{n-1})]$, where $\lambda = \dfrac{2a_1 + a_2 + a_3 + a_4 + a_5}{2 - a_2 - a_3 - a_4 - a_5}$, and $\lambda < 1$.

So, $F[d(x_{n+1},x_n)] \le \lambda\, F[d(x_n,x_{n-1})] \le \lambda^2\, F[d(x_{n-1},x_{n-2})] \le \dots \le \lambda^n\, F[d(x_1,x_0)]$ …(3.1.4)

Now, for m > n, we have that;

$d(x_n,x_m) \le d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2})] + \dots + d(x_{m-1},x_m)$

By the non-decreasing and subadditive of F we have:

$F[d(x_n,x_m)] \le F[d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2})] + \dots + d(x_{m-1},x_m)]$

$\leq F[d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \ldots + d(x_{m-1},x_m)]$
$\leq \lambda^n F[d(x_1,x_0)] + \lambda^{n+1} F[d(x_1,x_0)] + \ldots + \lambda^{m-1}F[d(x_1,x_0)]$
$\leq \frac{\lambda^{m-1}}{1-\lambda} F[d(x_1,x_0)] \longrightarrow 0 \text{ as } m \longrightarrow \infty$

Hence, $\lim_{n,m\to\infty} d(x_n,x_m) = 0$ by (2). Applying lemma (2.14), $\langle x_n \rangle$ is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

Let $c \in int(P)$ be given. We can choose $n \in N$ such that

$d(x_n,x_{n-1}) \ll F^{-1}\left(\frac{1-a_2-a_4}{a_3}\cdot\frac{C}{3}\right)$, $d(x_n,z) \ll F^{-1}\left(\frac{1-a_2-a_4}{a_5+1}\cdot\frac{C}{3}\right)$, and

$d(x_{n-1},z) \ll F^{-1}\left(\frac{1-a_2-a_4}{a_1+a_4}\cdot\frac{C}{3}\right)$ for all $n \in N$

By $\ll$-increasing and surjective of F, we have:

$F[d(x_n,x_{n-1})] \ll \left(\frac{1-a_2-a_4}{a_3}\cdot\frac{C}{3}\right)$, $F[d(x_n,z)] \ll \left(\frac{1-a_2-a_4}{a_5+1}\cdot\frac{C}{3}\right)$, and

$F[d(x_{n-1},z)] \ll \left(\frac{1-a_2-a_4}{a_1+a_4}\cdot\frac{C}{3}\right)$

Then we have by non-decreasing and subadditive of F:

$F[d(Tz,z)] \leq F[d(Tz,Tx_{n-1}) + d(Tx_{n-1},z)]$
$\leq F[d(Tz,Tx_{n+1})] + F[d(Tx_{n+1},z)]$
$\leq a_1F[d(z,x_{n-1})] + a_2F[d(Tz,z)] + a_3F[d(Tx_{n-1},x_{n-1})] + a_4F[d(Tz,x_{n-1})] + a_5F[d(Tx_{n-1},z)] + F[d(Tx_{n-1},z)]$
$\leq a_1F[d(z,x_{n-1})] + a_2F[d(Tz,z)] + a_3F[d(x_n,x_{n-1})] + a_4F[d(Tz,x_{n-1})] + a_5F[d(x_n,z)] + F[d(x_n,z)]$
$\leq a_1F[d(z,x_{n-1})] + a_2F[d(Tz,z)] + a_3F[d(x_n,x_{n-1})] + a_4F[d(Tz,z)] + a_4F[d(z,x_{n-1})] + a_5F[d(x_n,z)] + F[d(x_n,z)]$

$(1 - a_2 - a_4) F[d(Tz,z)] \leq (a_1 + a_4)F[d(z,x_{n-1}) + a_3F[d(x_n,x_{n-1})] + (a_5 + 1)F[d(x_n,z)]$

$F[d(Tz,z)] \leq \left(\frac{a_1+a_4}{1-a_2-a_4}\right)F[d(z,x_{n-1}) + \left(\frac{a_3}{1-a_2-a_4}\right) F[d(x_n,x_{n-1})] + \left(\frac{a_5+1}{1-a_2-a_4}\right)F[d(x_n,z)]$

$$\ll \frac{C}{3}+\frac{C}{3}+\frac{C}{3}=C$$

Thus, $F[d(Tz,z)] \ll \frac{C}{m}$ for all $m \in N$, and so $\frac{C}{m} - F[d(Tz,z)] \in P$ since $\frac{C}{m} \longrightarrow 0$ (as $m \longrightarrow \infty$) and P is closed, $-F[d(Tz,z)] \in P$, but $F[d(Tz,z)] \in P$, hence $F[d(Tz,z)] = 0$, by (F_1), $d(Tz,z) = 0$ and so $z = Tz$.

For the uniqueness of fixed point of T, suppose that u is another fixed point of T, then from equation (3.1.1), we have:

$F[d(z,u)] = F[d(Tz,Tu)]$

$\leq a_1F[d(z,u)] + a_2F[d(Tz,z)] + a_3F[d(Tu,u)] + a_4F[d(Tz,u)] + a_5F[d(Tu,z)]$

$\leq a_1F[d(z,u)] + a_2F[d(z,z)] + a_3F[d(u,u)] + a_4F[d(z,u)] + a_5F[d(u,z)]$

$\leq (a_1 + a_4 + a_5)\, F[d(z,u)]$

$\leq (a_1 + a_2 + a_3 + a_4 + a_5)\, F[d(z,u)]$

So, that implies;

$(1 - (a_1 + a_2 + a_3 + a_4 + a_5))\, F[d(z,u)] \leq 0.$

Since $\sum_{i=1}^{5} a_i < 1$ then $(1 - (a_1 + a_2 + a_3 + a_4 + a_5)) > 0$ therefore we have $F[d(z,u)] \leq 0$ and then $-F[d(z,u)] \in P$, but $F[d(z,u)] \in P$, and hence $F[d(z,u)] = 0$, by (F_1), $d(z,u) = 0$ and so $z = u$.

Therefore, T has a unique fixed point.

As a consequence of theorem (3.1), we have the following corollaries:

The first one is obtained which is the main result of Hardy and Rogers [6] in the setup of Cone metric space.

Corollary (3.2):

Let (X,d) be a complete Cone metric space. Suppose that a mapping $T:X \longrightarrow X$ satisfies:

$d(Tx,Ty) \leq a_1d(x,y) + a_2d(Tx,x) + a_3d(Ty,y) + a_4d(Tx,y) + a_5d(Ty,x)$

for all $x, y \in X$, where a_i, i = 1,2,3,4,5 are all non-negative constants such that $\sum_{i=1}^{5} a_i < 1$, then T has a unique fixed point and moreover, the iterative sequence $\langle T^n x \rangle$ is convergent to the fixed point.

Proof:

Taking F(a) = a for all a ∈ P in equation (3.1.1) of theorem (3.1), we obtain the required result.

Corollary (3.3):

Let (X,d) be a complete Cone metric space. Suppose that F ∈ $\tilde{U}(p,p)$ such that:

(i) F is non-decreasing and subadditive.

(ii) If, for $<c_n> \subset P$, $\lim_{n\to\infty} F(c_n) = 0$ then $\lim_{n\to\infty} c_n = 0$.

And let $T:X \longrightarrow X$ satisfies for all x, y ∈ X:

(1) $F[d(Tx,Ty)] \le a_2\{F[d(Tx,x)] + F[d(Ty,y)]\}$, where $a_2 \in [0,\frac{1}{2})$.

(This corollary is theorem (2.1) in [3]).

(2) $F[d(Tx,Ty)] \le a_4\{F[d(Tx,y)] + F[d(Ty,x)]\}$, where $a_4 \in [0,\frac{1}{2})$.

(This corollary is theorem (2.2) in [3]).

(3) $F[d(Tx,Ty)] \le a_1F[d(x,y)] + a_5F[d(Ty,x)]$, where a_1, a_5 are non-negative constants with $a_1 + a_5 < 1$.

(This corollary is theorem (2.3) in [3]).

(4) $F[d(Tx,Ty)] \le a_1F[d(x,y)] + a_4F[d(Tx,y)]$, where a_1, a_4 are non-negative constants with $a_1 + a_4 < 1$.

(5) $F[d(Tx,Ty)] \le a_1F[d(x,y)] + a_2F[d(Tx,x)]$, where a_1, a_2 are non-negative constants with $a_1 + a_2 < 1$.

(6) $F[d(Tx,Ty)] \le a_1F[d(x,y)] + a_3F[d(Ty,y)]$, where a_1, a_3 are non-negative constants with $a_1 + a_3 < 1$.

(7) $F[d(Tx,Ty)] \le a1F[d(x,y)] + a_2F[d(Tx,x)] + a_3F[d(Ty,y)]$, where a_1, a_2, a_3 are non-negative constants with $a_1 + a_2 + a_3 < 1$.

(8) $F[d(Tx,Ty)] \le a_1F[d(x,y)] + a_4F[d(Tx,y)] + a_5F[d(Ty,x)]$, where a_1, a_4, a_5 are non-negative constants with $a_1 + a_4 + a_5 < 1$.

(9) $F[d(Tx,Ty)] \le a_1F[d(x,y)]$, where $a_1 \in [0,1)$.

Then in all these cases, T has a unique fixed point and moreover the iterative sequence $<T^nx>$ is convergent to the fixed point.

Proof:

(1) By taking $a_1 = a_4 = a_5 = 0$ and $a_2 = a_3$ in equation (3.1.1) of theorem (3.1), we obtain the required result and you can see the proof of theorem (2.1) in [3].

(2) By taking $a_1 = a_2 = a_3 = 0$ and $a_4 = a_5$ in equation (3.1.1) of theorem (3.1), we obtain the required result, and you can see the proof of theorem (2.2) in [3].

(3) By taking $a_2 = a_3 = a_4 = 0$ in equation (3.1.1) of theorem (3.1), we obtain the required result, and you can see the proof of theorem (2.3) in [3].

(4) By taking $a_2 = a_3 = a_5 = 0$ in equation (3.1.1) of theorem (3.1), we obtain the required result.

(5) By taking $a_3 = a_4 = a_5 = 0$ in equation (3.1.1) of theorem (3.1), we obtain the required result.

(6) By taking $a_2 = a_4 = a_5 = 0$ in equation (3.1.1) of theorem (3.1), we obtain the required result.

(7) By taking $a_4 = a_5 = 0$ in equation (3.1.1) of theorem (3.1), we obtain the required result.

(8) By taking $a_2 = a_3 = 0$ in equation (3.1.1) of theorem (3.1), we obtain the required result.

(9) By taking $a_2 = a_3 = a_4 = a_5 = 0$ in equation (3.1.1) of theorem (3.1), we obtain the required result.

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