Approximation of Unbounded Functions
by Comonotone Polynomials

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ABSTRACT
The Purpose of this paper is to study the approximation of unbounded functions in the spaces $L_{p,\alpha}[-1,1]$, $0 < p \leq \infty$ by comonotone polynomials. We find the degree of best approximation of unbounded functions in terms of the second Ditizian-Totik modulus of smoothness.

1. INTRODUCTION

In recent years the approximation by comonotone polynomial has been studied by [1] such that R.K.Beston and D.Leviatan obtain some results on the approximation of comonotone polynomials and N.M.Kassim [2] discussed approximation of bounded function by using comonotone and monotone polynomials in $L_p$-spaces ($0 < p < 1$) in terms modulus of smoothness. The main departure from these previous works is that we shall prove direct estimates for the error of polynomial approximation in terms of the Ditizian Totik modulus of smoothness.

2. Definition and notation
Let $P_n$ denote the set of all algebraic polynomials of degree $\leq n$ and $L_{p,\alpha}[-1,1]$ the set of all functions on $[-1,1]$ such that,

$$
\|f\|_{p,\alpha} = \left( \int_{-1}^{1} |f(x)|^p e^{-\alpha x^2} \, dx \right)^{1/p} < \infty, \quad 0 < p < \infty
$$

Let $Y_r : = \{y_1, \ldots, y_r\}$, $y_0: = -1 < y_1 < \ldots < y_r < 1 =: y_{r+1}$, $r \in \mathbb{N}$. Denote by $\Delta^1(Y_r)$ the set of all nondecreasing functions $f$ on $[y_{r-2k}, y_{r-2k+1}]$ and is nonincreasing on $[y_{r-2k-1}, y_{r-2k}]$ ($k \in \mathbb{N}$) that is mean those have $r$ monotonicity changes at the points in $Y_r$ and are nondecreasing near 1.

Let $\Delta^1: = \Delta^1(Y_0)$ denote the set of all nondecreasing functions on $[-1,1]$. Function from the class $\Delta^1(Y_r)$ are said to be comonotone with one another, comonotone polynomial approximation is the approximation of function $f \in \Delta^1(Y_r)$ by polynomial which are comonotone with it.
For \( f \in L_{p,\alpha}[-1,1] \) let us define the degree of comonotone polynomial approximation of \( f \) by
\[
E_n^{(1)}(f, Y_p, p, \alpha) := \inf_{p_n \in P(Y)} \| f - p_n \|_{p, \alpha}, \text{ if exist}
\] ...(2.1)

The m-th order Ditizian-Totik modulus of smoothness \( \omega_m^0(f, \delta)_{p, \alpha} \) is given by
\[
\omega_m^0(f, \delta)_{p, \alpha} := \sup_{0 < h \leq \delta} \| \Delta_{\varphi(h)}^m(f, \cdot) e^{-\alpha x} \|_p,
\]
where \( \varphi(x) = \sqrt{1 - x^2} \) and
\[
\Delta_{\varphi(h)}^m(f, x) := \begin{cases} 
\sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} f(x - \frac{m}{2} \eta + i \eta), & x + \frac{m}{2} \eta \in [-1,1] \\
0, & \text{o.w}
\end{cases}
\]

Define \( d(r) \) by,
\[
d(r) := \min \{ y_1, y_2 - y_1, \ldots, y_r - y_{r-1}, 1 - y_r \}
\]
Now, in [3] for sufficiently large \( \mu = \mu(r) \), there exists polynomials \( v_n(x) \) and \( w_n(x) \) of degree \( \leq c(r) n \) such that the polynomial
\[
p_n(x) := (q_n(x) - q_n(y_1)) v_n(x) + q_n(y_1) w_n(x)
\]...(2.2)
is comonotone with \( f \), and the following inequalities are satisfied
\[
|\text{sgn}(x - y_j) - v_n(x)| \leq c(r) \psi_j^{\mu}(x),
\]
\[
|\text{sgn}(x - y_j) - w_n(x)| \leq c(r) \psi_j^{\mu}(x),
\]
Where \( \psi_j^{\mu}(x) = \frac{h_j}{|x - x_j| + h_j} \) (recall that \( y_1 \in [x_j, x_{j+1}] \)).

3-The Main Results

It is found that the degree of approximation of a comonotone polynomial are in stages. In the first approximant \( f \in L_{p,\alpha}[-1,1] \cap \Delta^1(Y_r) \) by continuous piecewise linear spline \( s \in \Delta^1(Y_r) \) that is
\[
\| f - s \|_{p, \alpha} \leq c \omega_2^0(f, n^{-1})_{p, \alpha},
\]
where
\[
\omega_2^0(f, n^{-1}) := \sup_{0 < h \leq \delta} \| \Delta_{\varphi(h)}^2(f, \cdot) e^{-\alpha x} \|_p
\]

Then the study shows how to approximate \( s \) by a polynomial in \( \Delta^1(Y_r) \), use [4] for formation the partition of the interval \([-1,1]\) by the nodes \( x_k, k = 0, \ldots, n \) with \( Y_r \), then delete \( x_i \) and \( x_{i-1} \) for which there is a \( y_j, j = 1, \ldots, r \) such that \( x_{i-1} \leq y_j \leq x_i \) and end up a new partition which denote \( Z_{r,n} \) by:
\( Z_{r,n} := Y_r \cap \left( \{ x_k \}_{k=0}^n \ \backslash \{ x_i, x_{i-1} : x_{i-1} \leq y_j \leq x_i \ \text{for some} \ j = 1, \ldots, r \} \right) \quad (3.1) \)

And can be chose \( \bar{\pi} \) for every interval \([-1,1]\) of the partition \( Z_{r,n} \) the restriction of \( \bar{\pi} \) to \( I \) is a near-best linear approximant to \( f \) in \( L_{p,\alpha}(I) \).

Let \( \bar{y} \in [x_{i-1}, x_i] \), say and suppose that \( p_1 \) is non decreasing and \( p_2 \) is non increasing.

Now define \( s \) in \([x_{i-3}, x_{i-2}]\) as the piecewise linear continuous spline
\[
s(x) = \begin{cases} 
 1 & \text{for } x \not\in (x_{i-2}, x_{i+1}) \\
 2 & \text{if } p_2(x_{i-2}) > p_1(x_{i+1}) 
\end{cases}
\]

A continuous piecewise linear spline \( s \in \Delta^1(Y_r) \) is obtained \([4]\).

**Lemma (3.1):**

Let a function \( f \in L_{p,\alpha} \cap \Delta^1(Y_r), \ 0 < p \leq \infty \), then for every \( n \geq c(r)/d(r) \) (\( d(r) \neq 0 \))there exists a continuous piecewise linear spline \( s \in \Delta^1(Y_r) \) on the knot sequence \( Z_{r,n} \) satisfying
\[
\| f - s \|_{p,\alpha} \leq c_2 \omega_2^0(f, n^{-1})_{p,\alpha} \quad \text{...(3.2)}
\]
and
\[
\omega_2^0(s, n^{-1})_{p,\alpha} \leq c_2 \omega_2^0(f, n^{-1})_{p,\alpha} \quad \text{...(3.3)}
\]

**Proof:**

First proof (3.3) by using (3.2)
\[
\omega_2^0(s, n^{-1})_{p,\alpha} = \omega_2^0(s - f + f, n^{-1})_{p,\alpha} \\
\leq c_1 \omega_2^0(s - f, n^{-1})_{p,\alpha} + c_2 \omega_2^0(f, n^{-1})_{p,\alpha} \\
\omega_2^0(s, n^{-1})_{p,\alpha} = c \| s - f \|_{p,\alpha} + c_2 \omega_2^0(f, n^{-1})_{p,\alpha} \\
\leq c_3 \omega_2^0(f, n^{-1})_{p,\alpha} + c_2(r) \omega_2^0(f, n^{-1})_{p,\alpha}
\]

Hence,
\[
\omega_2^0(s, n^{-1})_{p,\alpha} \leq c \omega_2^0(f, n^{-1})_{p,\alpha}
\]

Now to prove (3.2) construct a spline \( \bar{s} \) which satisfies the condition for each \( n > c(r)/d(r) \) (\( d(r) \neq 0 \))
\[
E_n^{(1)}(f, Y_r)_{p,\alpha} \leq c(r) \omega_2^0(f, n^{-1})_{p,\alpha}
\]

Consider the case where \( p_2(x_{i-2}) \leq p_1(x_{i+1}) \), the other case is analogous, all are shown as follows:
\[
\| f - s \|_{p,\alpha(x_{i-2}, y]} = \| f - p_2 + p_2 - s \|_{p,\alpha(x_{i-2}, y]} \\
\| f - s \|_{p,\alpha(x_{i-2}, y]} \leq \| f - p_2 \|_{p,\alpha(x_{i-2}, y]} + \| s - p_2 \|_{p,\alpha(x_{i-2}, y]}
\]

To estimate the term \( \| s - p_2 \|_{p,\alpha(x_{i-2}, y]} \). Indeed
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\[ \| s - p_2 \|_{p, \alpha(x_{i-2}, y)} = |s(y) - p_2(y) - p_1(y) - p_2(y)| \]
\[ \leq \| s - p_2 \|_{p, \alpha(x_{i-2}, y)} = |s(y) - p_2(y) - p_1(y) + p_2(y)| \]
\[ = \| s - p_2 \|_{p, \alpha(x_{i-2}, y)} = |s(y) - p_1(y)| \]
\[ \| s - p_2 \|_{p, \alpha(x_{i-2}, y)} = |s(y) - p_2(y) - p_1(y) - p_2(y)| \]
\[ \leq c |s(y) - p_2 - s(y) + p_1(y)| \]
\[ = c \| p_1 - p_2 \|_{p, \alpha(x_{i-2}, y)} \]

Therefore,
\[ \| s - p_2 \|_{p, \alpha(x_{i-2}, y)} = |s(y) - p_2(y) - p_1(y) - p_2(y)| \]
\[ \leq ch_i^{-1/p} \| p_1 - p_2 \|_{p, \alpha(x_{i-2}, y)} \leq c_1 \| p_1 - p_2 \|_{p, \alpha(x_{i-2}, y)} \]
where \( h_i = |x_{i-1} - x_i| \)

Then from (3.4)
\[ \| f - s \|_{p, \alpha(x_{i-2}, y)} \leq \| f - p_2 \|_{p, \alpha(x_{i-2}, y)} + \| s - p_2 \|_{p, \alpha(x_{i-2}, y)} \]
\[ \leq \| f - p_1 \|_{p, \alpha(x_{i-2}, y)} + \| p_1 - p_2 \|_{p, \alpha(x_{i-2}, y)} \]
\[ \leq \| f - p_1 \|_{p, \alpha(x_{i-2}, y)} + \| f - f + p_1 - p_2 \|_{p, \alpha(x_{i-2}, y)} \]
\[ \leq \| f - p_1 \|_{p, \alpha(x_{i-2}, y)} + c_1 \| f - p_1 \|_{p, \alpha(x_{i-2}, y)} + c_2 \| f - p_2 \|_{p, \alpha(x_{i-2}, y)} \]

Hence
\[ \| f - s \|_{p, \alpha(x_{i-2}, y)} \leq c \| f - p_1 \|_{p, \alpha(x_{i-2}, y)} + \| f - p_2 \|_{p, \alpha(x_{i-2}, y)} \]

Now in this part, prove that \( f \) is going to be a continuous piecewise linear function on the Knot sequence \( Z_{r,n} \) which belongs to \( \Delta^1(Y_r) \) and satisfying \( f(y_1) = 0 \).

**Lemma (3.2):**
Let \( y_1 \in I_j = [x_{j-1}, x_j] \) and set \( h_j : = |I_j| = x_j - x_{j-1} \). Show that
\[ \| f \|_{L_{p, \alpha(y_1, h_j, h_{j+1})}} \leq c_{\omega_2}(f, h_j, J_j)_{p, \alpha} \]
where
\( J_j = [x_{j-2}, x_{j+2}] \)

**Proof:**
In the first, take \( L \) to be the straight line such that \( L_{[x_{j-2}, y_1]} = f \mid_{[x_{j-2}, y_1]} \), and get

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Similarly, can show that

\[ \| f \|_{p, a[y_1, h_j + y_1]} \leq c_{w2}(f, h_j, I_j)_{p, a} \]

And (3.5) become,

\[ \| f \|_{p, a[y_1, h_j + y_1]} \leq c_{w2}(f, h_j, J_j)_{p, a} \]

Then

\[ \| f \|_{p, a[y_1, h_j + y_1]} \leq c_{w2}(f, h_j, J_j)_{p, a} \] \hspace{1cm} \cdots(3.5) \]

And, while \( I_j \subseteq J_j \) have \( |I_j| \leq c |J_j| - ch_j \), and for \( n \geq c(r) / d(r) \), with sufficiently large \( c(r) \) the result is:

\[ \omega_2(f, h_j, J_j)_{p, a} \leq c \omega_2^{o}(f, n^{-1})_{p, a} \] \hspace{1cm} (3.6) \]

And (3.5) become,

\[ \| f \|_{L_{p, a}} \leq c_{w2}^{o}(f, h_j, J_j)_{p, a} \]

Now, to prove the following theorem the following notations are necessary.

Define the function \( f(x) \) by:

\[ f(x) := \begin{cases} 
-f(x) & \text{if } x < y_i \\
 f(x) & \text{if } x \geq y_i 
\end{cases} \]
The function $\hat{f}$ is continuous piecewise linear spline from the class $\Delta^1(Y_r\{y_1\})$ and (3.6) implies that for $n \geq \frac{c(r)}{d(r)}$

$$c \omega_2^p(\hat{f}, n^{-1})_{p, \alpha} \leq \omega_2^p(f, n^{-1})_{p, \alpha}$$ ... (3.7)

**Lemma (3.3):**

Let $q_n$ be a polynomial which is comonotone with $\hat{f}$ such that $\|\hat{f} - q_n\|_p \leq c \omega_2^p(\hat{f}, n^{-1})_p$.

**Lemma (3.4):**

Let $f \in L_{p, \alpha}[−1, 1], (0 < p \leq \infty)$ and $q_n$ be comonotone polynomial we have

$$\|\hat{f} - q_n\|_{p, \alpha} \leq c \omega_2^p(\hat{f}, n^{-1})_{p, \alpha}$$

**Proof:**

$$\|\hat{f} - q_n\|_{p, \alpha} = \left\| (\hat{f} - q_n) e^{-\alpha x} \right\|_p$$

$$= \left\| \hat{f} e^{-\alpha x} - q_n e^{-\alpha x} \right\|_p$$

Let $\hat{f} e^{-\alpha x} = F$ and $q_n e^{-\alpha x} = p_n$ where $F$ is bounded function and $p_n$ comonotone polynomial then by using lemma (3.3),

$$\|\hat{f} e^{-\alpha x} - q_n e^{-\alpha x}\|_p = \|F - p_n\|_p$$

$$\leq c \omega_2^p(F, n^{-1})_p$$

$$= c \omega_2^p(\hat{f}, n^{-1})_{p, \alpha}$$

**Lemma (3.5):**

If $c \int_{-1}^1 \psi_j^{mp}(x) dx \leq c h_j$ and that $q_n$ is monotone near $y_1$ then

$$\int_{-1}^1 |q_n(y_1)e^{-\alpha x}|^p \psi_j^{mp}(x) dx \leq c \omega_2^p(f, n^{-1})_{p, \alpha}$$

**Proof:** by using (3.7) and lemma (3.4),

$$\int_{-1}^1 |q_n(y_1)e^{-\alpha x}|^p \psi_j^{mp}(x) dx \leq c h_j |q_n(y_1)e^{-\alpha x}|^p$$

$$\leq c \|q_n\|_{p, \alpha}^p$$

$$\leq c \|\hat{f} - \hat{f} + q_n\|_{p, \alpha}^p$$
\[ \leq \left\| \hat{f} - q_n \right\|_{p,\alpha}^p + c \left\| \hat{f} \right\|_{p,\alpha}^p \]
\[ \leq c \omega_2^q (f , n^{-1})_{p,\alpha}^p + c \left\| \hat{f} \right\|_{p,\alpha}^p \]
\[ \leq c \omega_2^q (f , n^{-1})_{p,\alpha}^p + c \omega_2^q (f , n^{-1})_{p,\alpha}^p \]

Hence
\[ \int_{-1}^1 \left| q_n (y_1) e^{-\alpha x} \right|^p \phi_j^{\mu p} (x) dx \leq c \omega_2^q (f , n^{-1})_{p,\alpha}^p \]

**Lemma (3.6):** [4]

Let \( i \in Z_{r,n}(y_1,x) \) and \( \hat{h}_i := z_{i-1} - z_i \) then
\[ \left| f' (z_{i+}) - f' (z_{i-}) \right| \leq c \hat{h}_i^{(p+1)/p} \omega_2^q (f , n^{-1})_p \]

**Lemma (3.7):**

Let \( f \in L_{p,\alpha}[-1,1] \) and \( 0 < p \leq \infty \) have
\[ \left| f' (z_{i+}) - f' (z_{i-}) \right| e^{-\alpha x} \leq c \hat{h}_i^{(p+1)/p} \omega_2^q (f , n^{-1})_{p,\alpha} \]

**Proof:**
\[ \left| f' (z_{i+}) - f' (z_{i-}) \right| e^{-\alpha x} = \left| f' (z_{i+}) e^{-\alpha x} - f' (z_{i-}) e^{-\alpha x} \right| \]
Let
\[ F'(z_{i+}) = f'(z_{i+} e^{-\alpha x}) \]
\[ F'(z_{i-}) = f'(z_{i-} e^{-\alpha x}) \]
where \( F' \) is bounded then by using lemma (3.6),
\[ \left| f' (z_{i+}) - f' (z_{i-}) \right| e^{-\alpha x} = \left| F'(z_{i+}) - F'(z_{i-}) \right| \]
\[ \leq c \hat{h}_i^{(p+1)/p} \omega_2^q (f , n^{-1})_p \]
\[ \leq c \hat{h}_i^{(p+1)/p} \omega_2^q (f , n^{-1})_{p,\alpha} \]

**Lemma (3.8):**

Let \( f \in \Delta^1 (Y_i) \) be a continuous piecewise linear spline on the \( Z_{r,n} \),
\( y_1 \in [x_{j-1},x_j] \) and \( f (y_1) = 0 \) then for all \( x \in [-1,1] \)
\[ \left| f (x) e^{-\alpha x} \right| \leq c \left( 1 + \frac{|x-x_j|}{\delta(x,x_j)} \right)^2 \delta_n (x,x_j) \frac{1}{\delta(x,x_j)} \omega_2^q (f , n^{-1})_{p,\alpha} \]

where
\[ \delta_n (x,x_j) = \min \{ \Delta_n (x), \Delta_n (x_j) \} \neq 0 \] for all \( i \)

**Proof:**
Set \( Z_{r,n} = \{ -1 = z_m < z_{m-1} < \ldots < z_i < z_0 = 1 \} \) and \( \hat{h}_i := z_{i-1} - z_i \), Fix \( x > y_1 \) (and similar case when \( x \leq y_1 \)) and denote \( Z_{r,n}(y_1,x) := \{ i \mid z_i \in Z_{r,n} : y_1 \leq z_i < x \} \).
Since \( f \) is piecewise linear, then
\[
|f(x) e^{-\alpha x}| = |(f(x) - f(y_1)) e^{-\alpha x}|
\]
\[
= |(f'(\zeta)) e^{-\alpha x}| (x - y_1), \text{ for some } \zeta \in (y_1, x)
\]
Now,
\[
\left| f'(\zeta) e^{-\alpha x} \right| \leq \left| (f'(\zeta) - f'(y_{1+})) e^{-\alpha x} \right| + \left| f'(y_{1+}) e^{-\alpha x} \right|
\]
\[
\leq \sum_{i \in \mathbb{Z}_{r,n}(y_1, \zeta)} \left| (f'(z_i) - f'(z_i)) e^{-\alpha x} \right|
\]
\[
\leq \sum_{i \in \mathbb{Z}_{r,n}(y_1, x)} \left| (f'(z_i) - f'(z_i)) e^{-\alpha x} \right|
\]
\[
\leq c \left( 1 + \frac{|x - x_j|}{\delta_n(x, x_j)} \right) \|x - x_j\|_{\mathbb{Z}_{r,n}(y_1, x)} \max_{i \in \mathbb{Z}_{r,n}(y_1, x)} \left| (f'(z_i) - f'(z_i)) e^{-\alpha x} \right|
\]
Since \( \hat{h}_i \geq \delta_n(x, x_j) \) then
\[
\hat{h}_i^{(\mathbb{Z}_{r,n}(y_1, x))} \geq \delta_n^{(\mathbb{Z}_{r,n}(y_1, x))} \hat{h}_i \]
and by lemma (3.7) get,
\[
\left| f(x) e^{-\alpha x} \right| \leq c \hat{h}_i^{(\mathbb{Z}_{r,n}(y_1, x))} \left( 1 + \frac{|x - x_j|}{\delta_n(x, x_j)} \right) \|x - x_j\|_{\mathbb{Z}_{r,n}(y_1, x)} \omega_2^\theta(f, n^{-1})_{p, \alpha}
\]
\[
\leq c \left( 1 + \frac{|x - x_j|}{\delta_n(x, x_j)} \right) \|x - x_j\| \delta_n^{(\mathbb{Z}_{r,n}(y_1, x))} \omega_2^\theta(f, n^{-1})_{p, \alpha}
\]
\[
= c \left( \frac{|x - x_j|}{\delta_n(x, x_j)} + \frac{|x - x_j|^2}{\delta_n(x, x_j)} \right) \delta_n^{(\mathbb{Z}_{r,n}(y_1, x))} \omega_2^\theta(f, n^{-1})_{p, \alpha}
\]
\[
= c \left( \frac{|x - x_j|}{\delta_n(x, x_j)} + \frac{|x - x_j|^2}{\delta_n(x, x_j)} \right) \delta_n^{(\mathbb{Z}_{r,n}(y_1, x))} \omega_2^\theta(f, n^{-1})_{p, \alpha}
\]
\[
\leq c \left( 1 + \frac{2|x - x_j|}{\delta_n(x, x_j)} + \frac{|x - x_j|^2}{\delta_n(x, x_j)} \right) \delta_n^{(\mathbb{Z}_{r,n}(y_1, x))} \omega_2^\theta(f, n^{-1})_{p, \alpha}
\]
\[
\leq 1 + \frac{|x - x_j|^2}{\delta_n(x, x_j)} \delta_n^{(\mathbb{Z}_{r,n}(y_1, x))} \omega_2^\theta(f, n^{-1})_{p, \alpha}
\]
Theorem (3.9):
Let \( f \in \Delta^{(1)}(Y_r) \) be a continuous piecewise spline then for each \( n \geq c(r) / d(r) \) there is a polynomial \( p_n \in P_n \) such that
\[
\|f - p_n\|_{p,\alpha} \leq c(r) \omega_2^p(f, n^{-1})_{p,\alpha}
\]

Proof:
From (2.2), (2.3) get,
\[
\|f - p_n\|_{p,\alpha}^p = \|f - (q_n(x) - q_n(y_1))v_n(x) + q_n(y_1)v_n(x) - q_n(y_1)w_n(x)\|_{p,\alpha}^p
\]
\[
= \|f - q_n(x)v_n(x) + q_n(y_1)v_n(x) - q_n(y_1)w_n(x)\|_{p,\alpha}^p
\]
\[
= \|\hat{f} - q_n\text{sgn}(x-y_1) + q_n\text{sgn}(x-y_1) - q_n(y_1)v_n(x) + q_n(y_1)v_n(x) - q_n(y_1)w_n(x)\|_{p,\alpha}^p
\]
\[
= \|\hat{f} - q_n\text{sgn}(x-y_1) + q_n\text{sgn}(x-y_1) - q_n(y_1)v_n(x) + q_n(y_1)v_n(x)
- q_n(y_1)w_n(x)\|_{p,\alpha}^p
\]
\[
= \|\hat{f} - q_n\text{sgn}(x-y_1)\|_{p,\alpha}^p + \|q_n\text{sgn}(x-y_1) - v_n(x)\|_{p,\alpha}^p + \|q_n(y_1)v_n(x) - w_n(x)\|_{p,\alpha}^p
\]
\[
\leq \|\hat{f} - q_n\text{sgn}(x-y_1)\|_{p,\alpha}^p + \|q_n\text{sgn}(x-y_1) - v_n(x)\|_{p,\alpha}^p + \|q_n(y_1)v_n(x) - w_n(x)\|_{p,\alpha}^p
\]
\[
\leq \|\hat{f} - q_n\text{sgn}(x-y_1)\|_{p,\alpha}^p + c\int_{-1}^{1}f(x)e^{-\alpha x}\left|\psi_j^p(x)\right|dx + c\int_{-1}^{1}q_n(y_1)e^{-\alpha x}\left|\psi_j^p(x)\right|dx
\]

Using lemmas (3.2), (3.4) and (3.8) we get,
\[
\|f - p_n\|_{p,\alpha}^p \leq \|\hat{f} - q_n\text{sgn}(x-y_1)\|_{p,\alpha}^p + c\int_{-1}^{1}f(x)e^{-\alpha x}\left|\psi_j^p(x)\right|dx + c\int_{-1}^{1}q_n(y_1)e^{-\alpha x}\left|\psi_j^p(x)\right|dx
\]
\[
\leq c\omega_2^p(f, n^{-1})_{p,\alpha}^p + c\int_{-1}^{1}\left(1 + \frac{|x-x_j|}{\delta_n(x,x_j)}\right)^{2p} \omega_2^p(f, n^{-1})_{p,\alpha}^p \psi_j^p(x)dx + c\omega_2^p(f, n^{-1})_{p,\alpha}^p
\]

Then,
\[
\|f - p_n\|_{p,\alpha}^p \leq c\omega_2^p(f, n^{-1})_{p,\alpha}^p
\]

Conclusions
Suppose that \( f \) is unbounded function and used moduli of smoothness to found equivalence relation between the degree of best approximation of this function and the moduli of smoothness in the weight \( L_{p,\alpha} \) space.
and found approximation by spline function by comonotone polynomial.

**REFERENCES**