Certain Type of Lie Algebra Action

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ABSTRACT

The main aim in this study is to look for an interesting action with new properties from a Schure’s lemma, which states that the action of the tensor product of Lie algebras representations has interesting property. Putting in mind that one of the two representations is usual and the other is the dual we obtain some results by relating the tensor product of dual representations with usual representations. Our main work here is to give a representation of Lie algebra by intertwine these actions (representations).

INTRODUCTION

A Lie algebra is a finite-dimensional vector space which is naturally endowed with a bilinear operation, Lie algebras can then be studied using purely algebraic tools, a part from the intrinsic interest, the theory of Lie algebras and its representations is used in various parts of mathematics, A representation of a group G is a homomorphism of the group G onto a group of linear map acting on vector space D, the tensor product of vector spaces D2 and D1 is a vector space D2 ⊗ D1, for which there exists bilinear map δ : D2×D1 → D2 ⊗ D1 which satisfies the following property. Whenever φ : D2×D1 → D is any bilinear map, there exists a unique linear map...
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θ : \mathcal{D}_2 \otimes \mathcal{D}_1 \rightarrow \mathcal{D}, such that θ \cdot \delta = \varphi, and this property is called universal property.[1], [2].

Our main interest in this thesis is to give a representation of Lie algebra by give the definition of Lie algebra action on tensor product and the natural action and intertwining these actions.

1. \textit{The action of Lie algebra on Hom. And tensor.}

\textit{Definition, (1.1), [3]:} Let \mathbb{A} be a non empty set and let \mathbb{G} be a group with neutral element \(e \in \mathbb{G}\), a left action of \(\mathbb{G}\) on \(\mathbb{A}\) is a map \(\varphi : \mathbb{G} \times \mathbb{A} \rightarrow \mathbb{A}\) such that satisfies the following \(\varphi(e, x) = x\) and \(\varphi(g, \varphi(h, x)) = \varphi(gh, x)\), for all \(x \in \mathbb{A}\) and \(g, h \in \mathbb{G}\).

\textit{Definition, (1.2), [4]:} Let \((\mathfrak{g}_1, [\cdot, \cdot])\) and \((\mathfrak{g}_2, [\cdot, \cdot])\) be two Lie algebras over the same field \(\mathcal{F}\), a homomorphism of Lie algebras from \(\mathfrak{g}_1\) into \(\mathfrak{g}_2\) is a linear map \(\Omega : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2\), such that
\[
\Omega[x, y] = \Omega(x) \cdot \Omega(y) - \Omega(y) \cdot \Omega(x),
\]
for all \(x, y \in \mathfrak{g}_1\), and for simply we denote it by \(\Omega[x, y] = \Omega(x)\Omega(y) - \Omega(y)\Omega(x)\), if in addition \(\Omega\) is a one-to-one and onto then \(\Omega\) is called a Lie algebra isomorphism. A Lie algebra isomorphism of a Lie algebra with itself is called a Lie algebra automorphism. A representation of the Lie algebra \(\mathfrak{g}\) is a (finite-dimensional) real or complex vector space \(\mathcal{D}\) together with a homomorphism \(\mathfrak{g} \rightarrow \text{gl}(\mathcal{D})\) of Lie algebra.

\textit{Definition, (1.3), [5]:} Let \(\rho_{N_1} : \mathfrak{g} \rightarrow \text{gl}(N_1)\) and \(\rho_{N_2} : \mathfrak{g} \rightarrow \text{gl}(N_2)\) be two representations of a Lie algebra \(\mathfrak{g}\), a homomorphism (or intertwining map) \(\psi \cdot (\rho_{N_1}(x)) = (\rho_{N_2}(x)) \cdot \psi\) is a linear map which commutes with the action of \(\mathfrak{g}\), a homomorphism \(\psi\) is said to be an isomorphism of representations if it is an isomorphism of vector space. If \(\psi\) is the intertwining map of representations and in addition is invertible then \(\psi\) said to be an equivalence of representations.

\textit{Definition, (1.4), [5]:} Let \(\mathfrak{g}_1\) and \(\mathfrak{g}_2\) be two Lie algebras and let \(\Omega_1\) and \(\Omega_2\) be representations of \(\mathfrak{g}_1\) and \(\mathfrak{g}_2\) acting on spaces \(N_1\) and \(N_2\) respectively. Then the tensor product of \(\Omega_1\) and \(\Omega_2\) denoted by \(\Omega_1 \otimes \Omega_2\) is a representation of \(\mathfrak{g}_1 \times \mathfrak{g}_2\) acting on \(N_1 \otimes N_2\) defined by:
\[ \Omega_1 \otimes \Omega_2(x, y) = \Omega_1(x) \otimes I + I \otimes \Omega_2(y) \] where \( I \) is the identity map, for all \( x \in \mathfrak{g}_1, y \in \mathfrak{g}_2 \).

**Example, (1.5):** Let \( \Omega_1 : \text{sl}(2, \mathbb{R}) \to \text{gl}(\mathbb{N}) \) be a representation of \( \text{sl}(2, \mathbb{R}) \) such that:

\[
\Omega_1(h) = a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \Omega_1(e) = b = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Omega_1(f) = c = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix};
\]

Where \( \{h, e, f\} \) is basis for \( \text{sl}(2, \mathbb{R}) \)

And let \( \{f_1, f_2, f_3\} \) be a basis for \( \text{so}(3) \), Where \( f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, f_2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, f_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \), with relations \([f_1, f_2] = f_3, [f_2, f_3] = f_1, [f_3, f_1] = f_2\).

And let \( \Omega_2 : \text{so}(3) \to \text{gl}(\mathbb{N}) \) be a representation, Such that: \( \Omega_2(f_1) = y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Omega_2(f_2) = y_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \Omega_2(f_3) = y_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \) then the tensor product of these representations is:

\[ \Omega_1 \otimes \Omega_2(h, f_1) = \Omega_1(h) \otimes I + I \otimes \Omega_2(f_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \]
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\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2
\end{pmatrix}_{9 \times 9}
\]

\textit{Definition, (1.6), [4]:} Let Ω be a representation of Lie algebra \( \hat{g} \) acting on a finite dimensional vector space \( \mathfrak{D} \), then the dual representation of \( \Omega \) is the representation of \( \hat{g} \) acting on \( \mathfrak{N}^* \) given by \( \Omega^*(x) = -(\Omega(x))^t \), the dual representation is also called contragradient representation.

2. The dual action of Lie algebra on Hom. and tensor.

\textit{Schure’s lemma, (2.1), [5]:} Suppose that \( \Omega_1 \) and \( \Omega_2 \) are representations of Lie algebra \( \hat{g} \) acting on finite-dimensional vector spaces \( \mathfrak{D}_1 \) and \( \mathfrak{D}_2 \), respectively.

Define an action of \( \hat{g} \) on \( \text{Hom}_F(\mathfrak{N}_2, \mathfrak{N}_1) \) by \( \psi : \hat{g} \rightarrow \text{gl}(\text{Hom}_F(\mathfrak{N}_2, \mathfrak{N}_1)) \), \( \psi(x)\theta = \Omega_1(x)\theta - \theta \Omega_2(x) \) for all \( x \in \hat{g} \) and \( \theta \in \text{Hom}_F(\mathfrak{N}_2, \mathfrak{N}_1) \) and:

\( \text{Hom}_F(\mathfrak{N}_2, \mathfrak{N}_1) \cong \mathfrak{N}_2^* \otimes \mathfrak{D}_1 \) as equivalence of representations.

\textit{Proposition, (2.2):} Let \( \Omega : \hat{g} \rightarrow \text{gl}(\mathfrak{D}) \) be a representation of Lie algebra \( \hat{g} \) on to \( F \)-finite dimensional vector space \( \mathfrak{D} \), then \( \Omega^* : \hat{g} \rightarrow \text{gl}(\mathfrak{D}^*) \) is the dual representation on \( \mathfrak{D}^* \) which is given by \( \Omega^*(x) = \rho^* \Omega(x) \), for all \( x \in \hat{g} \), where \( \rho : \mathfrak{D} \rightarrow F \).

\textit{proof}

Since \( \Omega \) is a linear map that satisfies \( \Omega([x,y]) = [\Omega(x), \Omega(y)] \)
then \( \Omega^* \) is a linear map that satisfies \( \Omega^*([x,y]) = [\Omega^*(x), \Omega^*(y)] \)
and for all basis \( \omega_j \in \mathfrak{D} \) there exist basis \( \omega^*_i \in \mathfrak{N}^* \) where \( i = 1, 2 ; j = 1, 2 \)
such that: \( \omega^*_i(\omega_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \) then \( \omega^*_i \) is called the dual basis of \( \omega_j \).

But in terms of matrices and linear map,

\( \text{gl}(\mathfrak{D}_2) \cong \text{gl}(m, \mathcal{F}) \) and \( \text{gl}(\mathfrak{D}_1) \cong \text{gl}(n, \mathcal{F}) \)

we have the matrix representations: \( \Omega_2 : \hat{g} \rightarrow \text{gl}(m, \mathcal{F}) \), \( \Omega_1 : \hat{g} \rightarrow \text{gl}(n, \mathcal{F}) \) and

\( \Omega_2^* : \hat{g} \rightarrow \text{gl}(m, \mathcal{F}) \) defined by \( \Omega_2^*(X) = -(\Omega_2(X))^t \)
\( \Omega_2^* \) is a Lie algebra homomorphism of Lie algebra \( \hat{g} \) into \( \text{gl}(m, \mathcal{F}) \) since:

\[
\Omega_2^*([X, Y]) = - (\Omega_2[X, Y])^t \\
= - ([\Omega_2(X), \Omega_2(Y)])^t \\
= - (\Omega_2(X)\Omega_2(Y) - \Omega_2(Y)\Omega_2(X))^t \\
= (\Omega_2(X))^t (\Omega_2(Y))^t - ((\Omega_2(Y))^t(\Omega_2(X))^t)) \\
= \begin{bmatrix} - (\Omega_2(X))^t & - (\Omega_2(Y))^t \end{bmatrix} \\
= [\Omega_2^*(X), \Omega_2^*(Y)], \text{ for all } X, Y \in \hat{g}.
\]

The following diagram shows that \( \Omega_2^* \) is a Lie algebra homomorphism.

**Figure 1**

**Remark, (2.3):** Let \( \Omega_1 \) and \( \Omega_2 \) be two representations of Lie algebras acting on vector spaces \( D_1, D_2 \) respectively, and let \( \text{Hom}_\mathcal{F}(D_2, D_1) \) be the \( \mathcal{F} \)-vector space of all linear maps from \( D_2 \) onto \( D_1 \), define \( \psi: \hat{g} \rightarrow \text{gl}(\text{Hom}_\mathcal{F}(D_2, D_1)) \) by:

\[
\psi(x)\theta = \Omega_1(x)\theta - \theta \Omega_2(x), \text{ for all } x \in \hat{g}, \theta \in \text{Hom}_\mathcal{F}(D_2, D_1) \\
(\psi(x)\theta)(v) = \Omega_1(x)\theta - \theta (\Omega_2(x)(v)) \text{ for all } x \in \hat{g}, \theta \in \text{Hom}_\mathcal{F}(D_2, D_1) \text{ and } v \in D_2.
\]

Thus \( \psi \) is a homomorphism of Lie algebras \( \hat{g} \) into \( \text{gl}(\text{Hom}_\mathcal{F}(D_2, D_1)) \).

The following diagram shows that the action of Lie algebra \( \hat{g} \) on \( \text{Hom}_\mathcal{F}(D_2, D_1) \) as follows:

\[
\psi(x)\theta = \Omega_1(x)\theta - \theta \Omega_2(x).
\]

**Figure 2**
Now to prove $\psi$ is a representation of Lie algebra, we must prove that $\psi$ is a homomorphism of Lie algebras $\hat{g}$ into $\text{gl}(\text{Hom}_F(\Omega_2, \Omega_1))$.

By definition of Lie algebra homomorphism we must prove that $\psi$ is a linear map and satisfy the following $\Omega([x, y]) = \Omega(x)\Omega(y) - \Omega(y)\Omega(x)$, for all $x, y \in \hat{g}$.

$\psi$ is a linear map since $\psi(\alpha x + \beta y) = \alpha \psi(x) + \beta \psi(y)$

Now to prove $\psi([x, y]) = \psi(x)(\psi(y)) - \psi(y)(\psi(x))$

The left hand side is $\psi([x, y]) = \Omega_1([x, y]) = \Omega_1(x)\Omega_1(y) - \Omega_1(y)\Omega_1(x)$

$\psi([x, y]) = \Omega_1(x)\Omega_1(y) - \Omega_1(y)\Omega_1(x)$

$\psi([x, y]) = \psi(x)(\psi(y)) - \psi(y)(\psi(x))$,

And the right hand side is:

$\psi(x)(\psi(y)) - \psi(y)(\psi(x)) = \Omega_1(x)(\psi(y)) - \Omega_1(y)(\psi(x))$

Now to prove $\psi([x, y]) = \psi(x)(\psi(y)) - \psi(y)(\psi(x))$

The following diagram shows that $\psi$ is A Lie algebra homomorphism.
Remark, (2.4): The map \( \psi \) above is called action of Lie algebra on 
\( \text{Hom}_F((D_2, D_1)) \).

Example, (2.5): Let \( \Omega_1 : \mathbb{R} \rightarrow \text{so}(3, \mathbb{R}) \subset \text{gl}(3, \mathbb{R}) \) and \( \Omega_2 : \mathbb{R} \rightarrow \text{o}(3, \mathbb{R}) \subset \text{gl}(3, \mathbb{R}) \) are two representations of Lie algebra \( \mathbb{R} \), where \( (n = m = 3) \) and \( D_1 \) is the \( \mathbb{R} \)-vector space of dim.3 and so is \( D_2 \), such that:

\[
\Omega_1(a) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -x \\ x & 0 & 0 \end{pmatrix}, \quad \Omega_2(a) = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ -x & 0 & 0 \end{pmatrix}, \quad \text{for all } a \in \mathbb{R}.
\]

The representation \( \psi \) of \( \mathfrak{h} \) on \( (D_2, D_1) \) is

\[
\psi : (D_2, D_1) \rightarrow \text{gl}(\text{Hom}_\mathbb{R}(D_2, D_1)) \cong \text{gl}(M(3, \mathbb{R}))
\]

such that

\[
(\psi(a)\vartheta)(v) = \Omega_1(a)\vartheta - \vartheta(\Omega_2(a)(v))
\]

\[
= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -x \\ x & 0 & 0 \end{pmatrix} \vartheta - \vartheta \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ -x & 0 & 0 \end{pmatrix}(v)
\]

\[
= \vartheta \begin{pmatrix} -0 & 0 & -x \\ 0 & 0 & 0 \\ 0 & x & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & -x \\ -x & x & 0 \end{pmatrix}.
\]

 Remark, (2.6): According to the remarks (3.1.3) and (3.1.4) that given above:

Let \( \psi : \mathfrak{g} \rightarrow \text{gl}(N_2^* \otimes N_1) \), which is a representation of \( \mathfrak{g} \) acting on the vector space \( N_2^* \otimes N_1 \) and defined by :

\[
\psi(x)(\omega_2 \otimes \omega_1) = (\Omega_2^*(x) \otimes 1 + I \otimes \Omega_1(x))(\omega_2 \otimes \omega_1)
\]

\[
\psi(x)(\omega_2 \otimes \omega_1) = \Omega_2^*(x)\omega_2 \otimes \omega_1 + \omega_2^* \otimes \Omega_1(x)\omega_1
\]

\[
(\psi(x)(\omega_2 \otimes \omega_1))(v) = \Omega_2^*(x)\omega_2^*(v) \otimes \omega_1 + \omega_2^*(v) \otimes \Omega_1(x)\omega_1
\]

Since \( (\omega_2 \otimes \omega_1) : N_2 \rightarrow N_1 \), then \( (\omega_2 \otimes \omega_1)(v) = \omega_2^*(v) \omega_1 \)

and we define \( \Omega_2^*(x)\omega_2^*(v) \otimes \omega_1 = -\omega_2^* (\Omega_2(x)(v)) \omega_1 \)

then the representation of Lie algebra on \( N_2^* \otimes N_1 \) becomes

\[
(\psi(x)(\omega_2 \otimes \omega_1))(v) = -\omega_2^*(\Omega_2(x)(v)) \omega_1 + \omega_2^*(v) \Omega_1(x)\omega_1
\]

for all \( x \in \mathfrak{g}, v \in D_2 \).

The following diagram shows that the action of Lie algebra \( \mathfrak{g} \) on tensor product as follows:

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ψ(x) (ω₂⁺ ⊗ ω₁) = (Ω₂⁺ (x) ⊗ 1 + I ⊗ Ω₁(x)) (ω₂⁺ ⊗ ω₁)

Figure 4

To show that ψ is a representation of ĝ acting on l(g(N₂⁺ ⊗ N₁)) where Ω₂ and Ω₁ are two representations of ĝ acting on N₂ and N₁ respectively.

ψ(x) (ω₂⁺ ⊗ ω₁) is a linear map since:
(ψ(α x + β y)) (ω₂⁺ ⊗ ω₁) = (α ψ(x))(ω₂⁺ ⊗ ω₁) + (β ψ(y))(ω₂⁺ ⊗ ω₁)
And:
ψ([x, y]) (ω₂⁺ ⊗ ω₁) = ψ(x)(ψ(y)(ω₂⁺ ⊗ ω₁)) - ψ(y)(ψ(x)(ω₂⁺ ⊗ ω₁))
thus, the left hand side is:
ψ([x, y]) (ω₂⁺ ⊗ ω₁) = (1 ⊗ Ω₁([x, y]))(ω₂⁺ ⊗ ω₁)
= ((Ω₂⁺(x)Ω₂⁺(y) - Ω₂⁺(y)Ω₂⁺(x)) ⊗ 1 + 1 ⊗ Ω₁(x)) = ((Ω₂⁺(x)Ω₂⁺(y) - Ω₂⁺(y)Ω₂⁺(x)) ⊗ 1 + 1 ⊗ Ω₁(x)Ω₁(y))
= (Ω₂⁺(x)(Ω₂⁺(y)ω₂⁺))ω₁ - (Ω₂⁺(y)(Ω₂⁺(x)ω₂⁺))ω₁ + ω₂⁺ ⊗ (Ω₁(x)Ω₁(y)ω₁)
= (Ω₂⁺(x)(Ω₂⁺(y)ω₂⁺))ω₁ + ω₂⁺ ⊗ (Ω₁(x)Ω₁(y)ω₁)

And the right hand side is ψ(x)(ψ(y)(ω₂⁺ ⊗ ω₁)) - ψ(y)(ψ(x)(ω₂⁺ ⊗ ω₁))
= Ω₂⁺(x)(ψ(y)ω₂⁺ ⊗ ω₁) + (ψ(y)ω₂⁺ ⊗ ω₁)Ω₁(x) - Ω₂⁺(y)(ψ(x)ω₂⁺ ⊗ ω₁)
= Ω₂⁺(x)(Ω₂⁺(y)ω₂⁺ ⊗ ω₁) + (Ω₂⁺(y)ω₂⁺ ⊗ ω₁)Ω₁(x) - (ψ(x)ω₂⁺ ⊗ ω₁)Ω₁(y)
= Ω₂⁺(x)(Ω₂⁺(y)ω₂⁺ ⊗ ω₁) + (Ω₂⁺(y)ω₂⁺ ⊗ ω₁)Ω₁(x) - (ψ(x)ω₂⁺ ⊗ ω₁)Ω₁(y)
= Ω₂⁺(x)(Ω₂⁺(y)ω₂⁺ ⊗ ω₁) + (Ω₂⁺(y)ω₂⁺ ⊗ ω₁)Ω₁(x) - (ψ(x)ω₂⁺ ⊗ ω₁)Ω₁(y)
= (Ω₂⁺(x)(Ω₂⁺(y)ω₂⁺)) ⊗ ω₁ + ω₂⁺ ⊗ (Ω₁(x)(Ω₁(y)ω₁))
Then the left hand side is equal to the right hand side, this means that
\( \psi([x,y]) (\omega_2 \otimes \omega_1) \) a Lie algebra homomorphism and then it is a
representation of Lie algebra.

Now if \( \psi \) is a matrix representation, then:
\[
\psi(X) (\omega_2 \otimes \omega_1) = (\Omega_2^* (X) \otimes I + I \otimes \Omega_1 (X)) (\omega_2 \otimes \omega_1)
\]
\[
\psi(X)(\omega_2 \otimes \omega_1) = \Omega_2^* (X) \omega_2 \otimes \omega_1 + \omega_2 \otimes \omega_1 \Omega_1 (X)
\]
\[
(\psi(X)(\omega_2 \otimes \omega_1))(v) = (\Omega_2^* (X) \omega_2^*)(v) \omega_1 + \omega_2^*(v) \Omega_1 (X) \omega_1
\]
\[
= -\omega_2^*(\Omega_2 (X)(v)) \omega_1 + \omega_2^*(v) \Omega_1 (X) \omega_1
\]
\[
= (\Omega_2^*(X) \otimes \Omega_1(X)) (\omega_2 \otimes \omega_1).
\]

**Proposition, (2.7):** Let \( \Omega_1 \) and \( \Omega_2 \) be representations of \( \hat{\mathfrak{g}} \) acting on \( \mathcal{F} \)-finite
dimensional vector spaces \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \), respectively. Then the action of lie
algebra of \( \hat{\mathfrak{g}} \) on \( \text{Hom}_{\mathcal{F}}(\mathcal{N}_2, \mathcal{N}_1) \) is equivalent to the action of Lie algebra \( \hat{\mathfrak{g}} \) on \( (\mathcal{N}_2^* \otimes \mathcal{N}_1) \).

**Proof:**
To show that \( \Phi : \mathcal{N}_2^* \times \mathcal{N}_1 \rightarrow \text{Hom}_{\mathcal{F}}(\mathcal{N}_2, \mathcal{N}_1) \) is a bilinear map.

let \( \Phi \) defined by:
\[
\Phi (\omega_2^*, \omega_1) = \theta, \text{ for all } \omega_2^* \in \mathcal{N}_2^*, \omega_1 \in \mathcal{D}_1,
\]
where \( \theta : \mathcal{N}_2 \rightarrow \mathcal{N}_1 \) is a linear map, defined by:
\[
\theta(v) = \omega_2^*(v) \omega_1
\]
for all \( \omega_2^*, \omega_2'^* \in \mathcal{N}_2^*, v \in \mathcal{N}_2, \alpha, \beta \in \mathcal{F}, \omega_1 \in \mathcal{N}_1 \)
\[
\Phi(\alpha \omega_2^* + \beta \omega_2'^*, \omega_1) = \left( (\alpha \omega_2^* + \beta \omega_2'^*) \right)(v) \omega_1
\]
\[
= \alpha \omega_2^*(v) \omega_1 + \beta \omega_2'^*(v) \omega_1
\]
\[
= \alpha \Phi(\omega_2^*, \omega_1) + \beta \Phi(\omega_2'^*, \omega_1).
\]

For all \( \omega_1, \omega_1' \in \mathcal{N}_1, \omega_2^* \in \mathcal{N}_2^* \)
\[
\Phi(\omega_2^*, \alpha \omega_1 + \beta \omega_1') = \omega_2^*(\alpha \omega_1 + \beta \omega_1')
\]
\[
= \omega_2^*(v)(\alpha \omega_1 + \beta \omega_1')
\]
\[
= \omega_2^*(v)(\alpha \omega_1) + \omega_2^*(v)(\beta \omega_1')
\]
\[
= \alpha \omega_2^*(v) \omega_1 + \beta \omega_2^*(v) \omega_1'
\]
\[
= \alpha \Phi(\omega_2^*, \omega_1) + \beta \Phi(\omega_2^*, \omega_1').
\]

So \( \Phi : \mathcal{N}_2^* \times \mathcal{N}_1 \rightarrow \text{Hom}_{\mathcal{F}}(\mathcal{N}_2, \mathcal{N}_1) \) is a bilinear map.

Thus by using the tensor and the universal property of this tensor product,
we get a unique linear map \( \theta \), see figure (5).

Such that for all \( \omega_2^* \in \mathcal{N}_2^*, \omega_1 \in \mathcal{D}_1 \) we have \( (\theta(\omega_2 \otimes \omega_1))(v) = \omega_2^*(v) \omega_1 \)
Figure 5
So, by the universal property of tensor product $N_2 \otimes N_1$, there exists a unique linear map $\theta : N_2^* \otimes N_1 \rightarrow \text{Hom}_F(N_2, N_1)$, this makes the above diagram commutative.

Consider the composition of linear maps, where $\omega_2^*(v)$ is defined as follows:
$$\theta(v) = \omega_1, \exists! f \in F \text{ such that } \omega_1 \rightarrow (f, \omega_1)$$
Since all maps are linear and $f$ is unique, put $\omega_2^*(v) = f$ related to $\omega_1$, see figure (6).

Figure 6
Define $\mathcal{L} : \text{Hom}_F(N_2, N_1) \rightarrow N_2^* \otimes N_1$ by $\mathcal{L}(\theta') = \omega_2^*(v)\omega_1$.
Define $\omega_2^* : N_2 \rightarrow F$, by $\omega_2^*(v) = f$, where $f$ is given by:
$$(\mathcal{L}(\theta'(v))) = (f, \theta'(v))$$, we can show that $\omega_2^*$ is linear.
Put $\theta'(v) = \omega_1$, for all $\omega_2^* \in N_2^*$, $\omega_1 \in D_1$, $\theta' \in \text{Hom}_F(N_2, N_1)$ and is related to $\omega_1$

$$\theta'(\alpha v_1 + \beta v_2) = \alpha \theta'(v_1) + \beta \theta'(v_2)$$
$$= \alpha f_1 + \beta f_2$$
$$= \alpha \omega_2^*(v_1) + \beta \omega_2^*(v_2), \text{ for all } v_1, v_2 \in D_2$$

Where:
$\omega_2^*(v_1) = f_1 \rightarrow \omega_2^*(\alpha v_1) = \alpha f_1$
$\omega_2^*(v_2) = f_2 \rightarrow \omega_2^*(\beta v_2) = \beta f_2$
$\omega_2^*(\alpha v_1 + \beta v_2) = \alpha f_1 + \beta f_2$
Clear $\theta'$ is a linear and $\mathcal{L}^{-1} = \theta$, thus $\mathcal{L}$ is a linear map.
Now to verify that $\theta$ is an intertwining map for the actions of $\hat{g}$ on $\mathbb{N}_2^* \otimes \mathbb{N}_1$ and on $\text{Hom}_F(\mathcal{D}_2, \mathcal{D}_1)$.

Let $\Omega_1$ and $\Omega_2$ be two representations of $\hat{g}$ acting on vector spaces $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively and consider the homomorphism $\Omega_2^* \otimes \Omega_1$ of Lie algebra $\hat{g}$ into the Lie algebra $\text{gl}(\mathbb{N}_2^* \otimes \mathbb{N}_1)$ defined by:

$((\Omega_2^* \otimes \Omega_1)(x))(\omega_2^* \otimes \omega_1) = (\Omega_2^*(x) \otimes I + I \otimes \Omega_1(x))(\omega_2^* \otimes \omega_1)$$= \Omega_2^*(x)\omega_2^* \otimes \omega_1 + \omega_2^* \otimes \omega_1 \Omega_1(x)$ where $x \in \hat{g}$, $\omega_2^* \in \mathbb{N}_2^*$, $\omega_1 \in \mathcal{D}_1$.

And apply the resulting map to the vector $v \in \mathcal{D}_2$ we get:

$(((\Omega_2^* \otimes \Omega_1)(x))(\omega_2^* \otimes \omega_1))(v) = -\omega_2^*(\Omega_2(x)(v))\omega_1 + \omega_2^*(v)\Omega_1(x)\omega_1$

Now let $\phi$ be the homomorphism of $\hat{g}$ into $\text{gl}(\text{Hom}_F(\mathcal{D}_2, \mathcal{D}_1))$ corresponding to $\Omega_2^* \otimes \Omega_1$ under the isomorphism $\theta$ for a map $\theta$ of the form (1).

$(\phi(x)\theta)(v) = \phi(x)\theta(\omega_2^* \otimes \omega_1)(v)$$= \theta((\Omega_2^* \otimes \Omega_1)(x)(\omega_2^* \otimes \omega_1))(v)$$= \theta(-\Omega_2(x))^t\omega_2^* \otimes \omega_1 + \omega_2^* \otimes \Omega_1(x)\omega_1(v)$$= -\omega_2^*(\Omega_2(x)(v))\omega_1 + \omega_2^*(v)\Omega_1(x)\omega_1$$= -\theta(\Omega_2(x)(v)) + \Omega_1(x)\theta(v)$.

Thus $\theta$ intertwines the actions of $\hat{g}$ on elements of the form $\omega_2^* \otimes \omega_1$.

Since every element of $\mathbb{N}_2^* \otimes \mathbb{N}_1$ is a linear combination of elements of this form, we conclude that $\theta$ is an intertwining map.

**REFERENCES**


Certain Type of Lie Algebra Action
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