On Local Rings

Zubayda M. Ibraheem ***Anees A. Fthee***
Department of Mathematics,
College of Computer Science and Mathematics
University of Mosul, IRAQ

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الملخص

تسمى الحلقة R بالحلقة المحلية اذا امتلكت مثالي أعظمي وحيد. في هذا البحث قدمنا بعض الخواص الأساسية لهذه الحلقة، كذلك درسنا العلاقة بين الحلقات المحلية والحلقات المنتظمة حسب مفهوم نيومان والحلقات المنتظمة بقوة .

ABSTRACT

A ring R is called local ring if it has exactly one maximal ideal . In this paper, we introduce some characterization and basic properties of this ring. Also ,we studied the relation between local rings and Von Neumann regular rings and strongly regular rings .

Keywords: *local, Von Neumann regular and strongly regular rings*

1. Introduction

Throughout this paper, R denotes associative rings with identity; all modules are unitary. For a subset X of R, the right (left) annihilator of X is denoted by r(X) (l(X)). If X={a}, we usually abbreviate it to r(a) (l(a)). We write J(R), Y(R), N(R) and U(R) for the Jacodson radical, right singular ideal, the set of all nilpotent elements of an R and the set of all invertible elements of an R respectively. A right R−module M is called P−injective, if for any principal right ideal aR of an R and any right R−homomorphism of an aR into an M can be extended to one of an R into an M. The ring R is called right P−injective if an R_R is P−injective [8]. An ideal I of a ring R is said to be essential if and only if I has a non−zero intersection with every non−zero ideal of an R. A ring R is reduced if N(R)=0. A ring R is said to be Von Neumann regular (or just regular) if and only if for each a in R there exists b in R such that a=aba [6]. A ring R is said to be right (left) quasi duo ring if every right (left) maximal ideal in R is right (left) ideal [7].

2. The Local Rings:

This section is devoted to give the definition of local rings with some of its chacterization and basic properties.

2.1. Definition [3] : A ring R is said to be a local ring ,if any one of the following conditions holds:

1) R has a unique maximal right ideal.
2) R has a unique maximal left ideal.
3) R/J(R) is a division ring.
4) R\U(R) is an ideal of R (all non−invertible elements of R form a proper ideal) i.e. J(R) is the set of all non−invertible elements of R.
5) R\U(R) is a group under addition.

6) For any n, $a_1+a_2+\ldots+a_n \in U(R)$ implies that some $a_i \in U(R)$
7) $a+b \in U(R)$ implies that $a \in U(R)$ or $b \in U(R)$.

2.2. Proposition [1] : For any non−zero ring R, the following statements are equivalent
1) R is a local ring.
2) if $a \in R$, then either a or 1-a is invertible.

Hazeinkel and Gubareni in [1] proved the following result:

2.3. Lemma [2] : Let R be a ring, all of whose non−invertible elements are nilpotent, then, R is a local ring.

As a consequence of this result and by using Proposition(2.3), we obtain the following result.

2.4. Proposition : Let an R be a local ring. Then, every element in an R is either invertible or nilpotent.

Proof : Suppose that an R is a local ring and let a be a non−invertible element in R, then by Proposition (2.2), 1-a is invertible, that is there exists u in R such that (1-a)u=1, and that can be held when $U=1+a+a^2+\ldots+a^{n-1} \in R$ and $(1-a)(1+a+a^2+\ldots+a^{n-1})=1$, but $1-a^n=(1-a)(1+a+a^2+\ldots+a^{n-1})=1$. Hence, $a^n=0$ and therefore, a is nilpotent element ■

2 .5. Corollary : Let an R be a local ring. Then, N(R)=J(R).

Proof : Let $0\neq a \in J(R)$. Then, a is non−invertible element. Thus, by Proposition (2.4) a is nilpotent element, that is $a \in N(R)$ Therefore, $J(R) \subset N(R)$. It is clear that $N(R)\subset J(R)$ [5]. Therefore, N(R)=J(R) ■

Remark: If an R is a local ring, then J(R) is nilideal.

In the next proposition, we give the necessary condition for R/I to be a local ring.

2.6. Proposition : Let R be a commutative ring and I be a primary ideal. Then, R/I is a local ring.

Proof : Suppose that $\bar{a}=(a+I)$ is for every $a \in R$.To prove that R/I is a local ring. It is enough to show that R/I has exactly two idempotent elements which are $\bar{0}$ and $\bar{1}$ [3]. Let $a\in R$ such that $\bar{a}$ be a non−zero idempotent element of R/I, we have $a^2-a \in I$. Since I is a primary ideal of an R and $a \notin I$, then there exists a non−negative integer n such that $(a-1)^n \in I$. By the binomial theorem (which is valid in any commutative ring), $(a-1)^n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} a^k \in I$. Now, we prove by mathematical induction; first we claim that for each $k \geq 2$, $a^k=a+x(1+a+\ldots+a^{k-2})$. Indeed; it is certainly true for k=2, that is $a^2=a+x$. Now, suppose the statement is true for k, then we get the following equalities $a^{k+1}=a^2+x(a+a^2+\ldots+a^{k-1})=a+x(1+a+\ldots+a^{k-2})$ we conclude that for each non−negative integers n, there is some $x_k \in I$ such that $a^k=a+x_k$. Now, $(-1)^n.1+\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}(a+x_k) \in I$. But, $(-1)^n.1+\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}a=(-1)^n(1-a)$. Hence, $1-a \in I$, and so $\bar{a}=\bar{1}$. Therefore, R/I is a local ring ■

Example : In (Z,+,.), 8Z is a primary ideal and Z/8Z is a local ring, where $Z/8Z = Z_8$ and has 2+8Z as an unique maximal ideal .

2.7. Lemma [5] : For any ring R the followings are equivalent:
1) R has an unique prime ideal.
2) R is a local ring and J(R) is the intersection of all prime ideals of R.
3) Every non invertible element is nilpotent.

Now, we can obtain the following result

2.8. Proposition: If an R is a commutative ring and has exactly one prime ideal, then R/N(R) is field .

Proof: Suppose an R has exactly one prime ideal, then by Lemma (2.7), R is a local ring and, hence R/J(R) is a division ring and by Proposition (2.5), we have J(R)=N(R) and since R is commutative therefore R/N(R) is field ■

Now, to show the relation between a local ring and Noetherain ring we have the following result from [1].

2.9. Lemma [1] : Suppose $\{I_i : i \in N\}$ is a family of proper right ideals of a ring R with property that $I_n \subset I_{n+1}$ for all $n \in N$. Then, $I=\bigcup_{n\in N} I_n$ is proper right ideal

2.10. Lemma [1] : Any proper right ideal I of a ring R with identity is contained in a maximal proper right ideal.

2.11. Proposition: Let an R be a local ring, then R is a Noetherain ring .

Proof: Let $\{I_i : i \in N\}$ be a family of proper ideals in the ring R. Since, an R is a local ring, then an R has a unique maximal ideal M. Then, by Lemma (2.10), the maximal ideal M contains all proper ideals in an R that is $I_i \subseteq M$, for all $i \in N$. Hence, $\bigcup_{i=1}^{n} I_i = I_n = M = J(R)$. Now , if $a \in I_i$ and $a \notin I_{i+1}$, then $I_i \not\subset I_{i+1}$. By Lemma(2.9) we get $\bigcup_{i=1}^{n} I_i \neq M$. which is contradiction. Hence, for all $a \in I_i$ we have $a \in I_{i+1}$ thus $I_i \subseteq I_{i+1}$, for all $i \in N$. That is $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots \subseteq I_n$. Hence, an R satisfies (a.c.c.). Therefore, R is a Noetherain ring ■

Note: The converse of the above proposition is not true, by the following example :

Example : (Z,+,.), where Z is the set of integer number and (Z,+,.) is a Noetherain ring, but not a local ring.

2.12. Proposition [1] : For any non−zero ring R the following statements are equivalent

1) R has a unique maximal right ideal.
2) R has a unique maximal left ideal.

2.13. Proposition : Let an R be a local ring. Then, R is quasi duo ring.

Proof: Let an R be a local ring .Then, an R has unique maximal ideal J(R). Now by Proposition (2.12), J(R) is two sided ideal. That is every right (left) maximal ideal in an R is ideal. Hence, an R is quasi duo ring ■

Note : The converse of the proposition (2.13) is not true.

Example: Let an R ={$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$: $a,b,c \in Z_2$ **}** an R is quasi duo ring, but not a local ring.

3.The Local Ring without Zero Divisor Element

In this section, we give some results about the local ring without zero−divisor and some relation with other rings like division ring ,Von Neumann regular, strongly regular and uniform rings.

3.1. Proposition: Let an R be a local ring. Then, every non−zero divisor element of R is a right invertible.

Proof: Let $0\neq a$ be a non zero−divisor element in R. Since, an R is a local ring, then either aR=R or aR ≠ R. If aR=R then, there exists b in R such that ab=1. Therefore, a is a right invertible. Now, if aR≠ R, then there exists a maximal right ideal M such that $aR \subset M$. Since, an R is a local ring, then $aR \subset M= J(R)$, which implies that $a \in J(R)$

and, hence by Proposition (2.5) a is nilpotent element. Therefore, a=0 which is a contradiction, therefore aR=R for all a in R and, hence a is invertible element ■

3.2. Proposition: Let R be a local ring without zero−divisor. Then R is regular ring.

Proof: Since, an R is a local ring, then either aR = R, that is there exists b∈R such that ab=1. Hence, $a^2b=a$. Thus, an R is a strongly regular ring and therefore, an R is a regular ring. Now, if aR ≠R, then by the same method of the proof of proposition (3.1) we have a=0 ■

3.3. Proposition:[2] For any ring R the followings are equivalent :

1) R is Von Neumann regular ring.
2) Every R−module is P−injective.
3) Every cyclic R−module is P−injective.

3.4. Proposition:[4] If an R is right P−injective, then J(R)=Y(R).

Now, we give the main result of this section:

3.5. Proposition: Let an R be a local ring and without zero divisor. Then

1) Y(R) = J(R).
2) Y(R) = N(R).

proof : 1)Since, an R is a local ring and without zero divisor, then by Proposition (3.2) and Proposition (3.3) R is right P−injective module and, hence by Proposition (3.4), we have Y(R) = J(R)

2) The proof is obvious ■

Finally, we give the following result :

3.6. Proposition: Let an R be a ring and without zero−divisor. Then, aR+r(a)=R if and only if R is a local ring**.**

Proof : Assume that an R is a local ring, then either aR=R for any element 0≠a in R so ar=1, where r∈R that is $a^2r = a$. Thus, a(1-ar)=0 implies that (1-ar)∈ r(a). Hence, 1=ar+(1-ar)∈ aR+r(a). Therefore, R=aR+r(a). Now, if aR≠R, then by the same proof Proposition (3.1), a=0. Now, suppose that R=aR+r(a), then there exists r ∈R,b∈ r(a) such that ar+b=1 implies that a^2r =a, then (1-ar) ∈ r(a)=0, that is 1= ar, thus a is right invertible. Now, since 1=ar. Then, a=ara implies that a-ara=0 , hence (1-ra) ∈ r(a)=0.Thus, a is left invertible. Therefore, a is a invertible .Now, we must prove that (1-a) is not invertible. If (1-a) is invertible, that is a ∈ J(R) and by the same method of the proof of Proposition (3.1) a=0 and this is contradiction, since a≠0. Thus, (1-a) is not invertible and, therefore by Proposition(2.2), an R is a local ring ■

REFERENCES

[1] Hazeinkel, M., Gubareni, N. and Kirichenlo, V.V. (2004), "Algebars, Rings and Modules" Vol. 1 kluwer Academic publishers.

[2] Ibraheem Z.M. (1991), "On P−injective modules" M.Sc. Thesis, Mosul University.

[3] Lam, T.Y., (1991), "A First Course in Noncommutative Rings" Springer−verlag New York, Inc.

[4] Nicholson, W. K. and Yousif M.F. (1995), "Principally Injective Rings", J. Algebra, 174, PP. 77−93.

[5] Naoum F.S. (2004), " On Semi Commutative π-Regular Rings", Ph.D. Thesis, Mosul University.

[6] Von Neumann, J. (1936), "On Regular rings", Proc. Nat. scad. Science U. S. A., Vol. 22, PP. 707−713.

[7] Yu, H.P., (1995), "On quasi duo ring", Glasgow Math. 37, PP. 21−31.

[8] Yue Chi Ming, R. (1974), "On Von Neumann regular rings", Proc, E Edinburgh Math. Soc. 19, PP. 89−91.