Characterization of Borel Soft Lattices

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Abstract

Soft set theory was introduced by Molodtsov in 1999 as a general mathematical for dealing with problems that contain uncertainty. In this paper, we define the concept of a soft latticeσ-algebra, measurable soft lattice, open soft sub lattice, open soft lattice, product Borel soft lattice and study their related properties.

المستخلص

نظرية المجموعة الناعمة قدمت من قبل Molodtsov في عام 1999 باعتبارها نظرية رياضية عامة للتعامل مع المشاكل التي تحتوي على عدم اليقين. في هذا البحث, نعرف مفهوم الحزم الجبرية الناعمة-σ , حزم القياسية الناعمة, الحزم الجزئية المفتوحة الناعمة, الحزم المفتوحة الناعمة و ضرب حزم بوريل الناعمة ودراسة الخصائص المرتبطة بها.

1 Introduction

Soft set theory was firstly introduced by Molodtsov in 1999 as a general mathematical tool for dealing with uncertainty. The operations of soft sets are defined by Maji et al. (2003) made soft set theory and redefined by Cağman and Enginoğlu(2010) have been studied Soft set theory and uni-int decision making. Recently, the properties and applications on the soft set theory have been studied increasingly[1,2,3,7,8].

The soft lattice structures are constructed by Nagarajan and Meenambigai [6] , Li [5] and Karaaslan et al. [4]. Over a soft lattice. In this paper, we deal with the algebraic structure on some collection of soft lattice, we define the notions of Borelsoft lattice, product Borel soft lattice, intersection (union) Borel soft lattice and measurable soft lattice. We focus on the algebraic properties of these notions.

Throughout this work, Urefers to an initial universe, $P(U)$ is the power set of U, F is a set of parameters and $A \subseteq F$.

2 Preliminaries

This section deals with the main definitions of Borel soft lattice and its properties which are included throughout the paper.

Definition 2.1: [4]

A function $f_A: F \to P(U)$ such that $f_A(x) = \phi$ if $x \notin A$, is called a soft set over U.

The set of all soft sets over U is denoted by $S(U)$.

Definition 2.2: [4]

Let $SL \subseteq S(U)$, and $\curlyvee$ and $\curlywedge$ be two binary operations on SL. If the set SL is equipped with two commutative and associative binary operations $\curlyvee$ and $\curlywedge$ which are connected by absorption law, then algebraic structure $(SL,\curlyvee,\curlywedge)$ is called a ***soft lattice***.

Theorem 2.3: [4]

Let $(SL,\curlyvee,\curlywedge)$ be a soft lattice and $f_A, f_B \in SL$. Then
$f_A \curlywedge f_B = f_A$ if and only if $f_A \curlyvee f_B = f_B$.

Definition 2.4:[4]

Let$(SL,\curlyvee,\curlywedge)$ be a soft lattice. If every subset of SL have both a greatest lower bound and a least upper bound, then it is called ***complete soft lattice***.

Definition 2.5:[4]

$(SL,\curlyvee,\curlywedge,\preccurlyeq)$ be a soft lattice and $E \subseteq SL$. If E is a soft lattice with the operations of SL, then E is called a ***soft sublattice***.

Definition 2.6:[4]

$(SL,\curlyvee,\curlywedge,\preccurlyeq)$ be a soft lattice. If SL satisfies the following axioms, it is called ***distributive soft lattice***:

$$f_A \curlywedge (f_B \curlyvee f_C) = (f_A \curlywedge f_B) \curlyvee (f_A \curlywedge f_C)$$
$$f_A \curlyvee (f_B \curlywedge f_C) = (f_A \curlyvee f_B) \curlywedge (f_A \curlyvee f_C)$$

For all f_A, f_B and $f_C \in SL$.

Definition2.7

IfSL is a soft lattice and satisfies the following conditions, then it is called a **soft lattice σ-algebra**:

- For all $f_h \in SL$, ${f_h}^c = f_{h^c} \in SL$
- If $f_{h_n} \in SL$ for $n = 1,2, \ldots$ then $\bigvee_{n=1}^{\infty} f_{h_n} \in SL$.

We denoted σSL, as the soft σ-algebra generated by SL.

Definition 2.8

A soft lattice $f_A$is said to be ***soft lattice measurable set*** if f_A belong to σSL.

Example:

The interval (a, ∞) is soft lattice measurable.

Definition 2.9

Let f_X and f_Y be two soft lattice then their Cartesian product denoted by $f_{X \times Y}$ is defined as $f_{X \times Y} = \{ f_{(x,y)} | f_x \in f_X , f_y \in f_Y \}$.

Definition2.10

For any f_a, f_b belongs to SL, we define an open sub lattice by $f_{(a,b)} = \{f_x | f_a \prec f_x \prec f_b\}$.

Definition 2.11

Union of countable number of open sub soft lattices is called an ***open soft lattice***.

3 Main Results

In this section, the notion of Borel soft lattices is introduced and several related properties and some characterization theorems are investigated.

Definition 3.1

The smallest σ-algebra containing class of all open soft lattices is called class of Borel soft lattices in R and it is denoted by $BSL(\beta)$. Every member of Borel class β are called ***Borel soft lattice*** of R .

Note

The family of measurable soft lattices is denoted by μSL.

Theorem 3.2

Every Borel soft lattice is a measurable soft lattice

Proof:

We known that measure of an interval is its length by Example. For any a belongs to R, (a,∞) is a measurable soft lattice and hence its complement $(-\infty, a]$ is also measurable soft lattice. Now, for any b belongs to R

$(-\infty, b) = \bigvee_{n=1}^{\infty}(-\infty, b-\frac{1}{n}]$ and since $(-\infty, b-\frac{1}{n}]$ is measurable soft lattice so is $(-\infty, b)$. Since $(a,b) = (-\infty, b) \wedge (a, \infty)$ it follows that each open soft sub lattice is measurable soft lattice.

Also we known that an open lattice is the union of countable number of open sub lattices we get every open soft lattice is a measurable soft lattice, but BSL is the smallest σ-algebra containing class of all open soft lattice in R, that is $BSL \preccurlyeq \mu SL$ implies every Borel soft lattice is measurable soft lattice.

Theorem 3.3

If f_A, f_B be a Borel soft lattice then $f_{A\times B}$ is also Borel soft lattice.

Proof:

Let f_A, f_B be a Borel soft lattice $BSL(\beta)$ and let $f_a \in f_A$ *and* $f_b \in f_B$ such that f_A, f_B is smallest σ-algebra ($f_A \prec BSL(\beta)$ and $f_B \prec BSL(\beta)$. Since f_a, f_b is a class of Borel, then

$f_a = \bigvee_{i=1}^{\infty} f_{a_i}$ *and* $f_b = \bigvee_{i=1}^{\infty} f_{b_i}$.

Therefore $f_{(a,b)} = (\bigvee_{i=1}^{\infty} f_{a_i}, \bigvee_{i=1}^{\infty} f_{b_i}) = \bigvee_{i=1}^{\infty} (f_{a_i}, f_{b_i}) \prec BSL(\beta)$.

But $f_A \prec BSL(\beta)$ and $f_B \prec BSL(\beta)$, hence $f_{(a,b)} = \bigvee_{i=1}^{\infty} (f_{a_i}, f_{b_i}) \in f_{A\times B}$. So $f_{A\times B}$ is a Borel soft lattice.

Theorem 3.4

If f_h, f_g be a Borel soft lattice then $f_{h\circ g}$ is also Borel soft lattice.

Proof:

If f_h, f_g be a Borel soft lattice $BSL(\beta)$. We known f_h, f_g is smallest σ-algebra such that $f_h(x) \prec \beta$ *and* $f_g(x) \prec \beta$ then $f_h = \bigvee_{n=1}^{\infty} f_{h_n}$ *and* $f_g = \bigvee_{n=1}^{\infty} f_{g_n}$

$$(f_{h\circ g})(x) = f_h(f_g)(x) = f_h\left(\bigvee_{n=1}^{\infty} f_{g_n}(x)\right) = \bigvee_{n=1}^{\infty} [f_{h_n(f_{g_n})}(x)]$$
$$= \bigvee_{n=1}^{\infty}\left[f_{h_n}(x)\circ f_{g_n}(x)\right] = \bigvee_{n=1}^{\infty}\left(f_{h_n\circ g_n}\right)(x)$$

Since every member of class is Borelsoft lattice and f_h, f_g is smallest σ-algebra. Then $f_{h\circ g}$ is smallest σ-algebra containing class of all open soft lattices, hence $f_{h\circ g}$ is also Borelsoft lattice.

Theorem 3.5

Let f_S be a Borel soft lattice then $f_S = \sum_{i=1}^{n} f_{S_i}$ is Borelsoft lattice.

Poof:

Since f_S be a Borel soft lattice β then f_{S_i} is also Borelsoft lattice for each $i = 1,2,\dots,n$. i.e. $f_S \prec \beta$ then $f_{S_i} \prec \beta$.

Let $f_{a_i} \in f_{S_i}$ be a class of Borel thus by definition $f_{S_i} = \bigvee_{i=1}^{\infty} f_{a_i}$. Therefore

$$f_S = \sum_{i=1}^{n} f_{s_i} = \sum_{i=1}^{n}\bigvee_{i=1}^{\infty} f_{a_i} = \bigvee_{i=1}^{\infty}(\sum_{i=1}^{n} f_{a_i})$$

But $f_{a_i} \in f_{S_i} \prec \beta$, then $\sum_{i=1}^{n} f_{a_i} \prec \beta$. Also $\bigvee_{i=1}^{\infty}(\sum_{i=1}^{n} f_{a_i}) \prec \beta$, then f_S is Borel soft lattice.

Corollary3.6
If f_{E_1}, f_{E_2} be a Borel soft lattice. Then $f_{E_1} \curlyvee f_{E_2} = f_{E_1 \curlyvee E_2}$ is a Borelsoft lattice.
Proof:
Let f_{E_1}, f_{E_2} be a Borel lattice βsuch that$f_{E_1} \prec \beta$and$f_{E_2} \prec \beta(f_{E_1}, f_{E_2}$ is smallest σ-algebra).
Let$f_{a_1} \in f_{E_1}$and$f_{b_1} \in f_{E_2}$such that$f_{E_1} = \bigvee_{i=1}^{\infty} f_{a_i}$ *and* $f_{E_2} = \bigvee_{i=1}^{\infty} f_{b_i}$ .Thus
$f_{E_1} \curlyvee f_{E_2} = \bigvee_{i=1}^{\infty} f_{a_i} \curlyvee \bigvee_{i=1}^{\infty} f_{b_i} = \bigvee_{i=1}^{\infty} (f_{a_i} \curlyvee f_{b_i}) \prec \beta.$
Therefore $f_{E_1} \curlyvee f_{E_2}$ is a Borel lattice.

Corollary3.7
If f_{E_1}, $_{E_2}$ be a Borel soft lattice. Then $f_{E_1} \curlywedge f_{E_2} = f_{E_1 \curlywedge E_2}$ is a Borel soft lattice.
Proof:
Let f_{E_1}, f_{E_2} be a Borel lattice βsuch that$f_{E_1} \prec \beta$and$f_{E_2} \prec \beta(f_{E_1}, f_{E_2}$ is smallest σ-algebra).
Let $f_{a_1} \in f_{E_1}$and$f_{b_1} \in f_{E_2}$s.t$f_{E_1} = \bigvee_{i=1}^{\infty} f_{a_i}$ *and* $f_{E_2} = \bigvee_{i=1}^{\infty} f_{b_i}$ .Thus
$f_{E_1} \curlywedge f_{E_2} = \bigvee_{i=1}^{\infty} f_{a_i} \curlywedge \bigvee_{i=1}^{\infty} f_{b_i} = \bigvee_{i=1}^{\infty} (f_{a_i} \curlywedge f_{b_i}) \prec \beta.$
Therefore $f_{E_1} \curlywedge f_{E_2}$ is a Borel lattice.

Corollary 3.8
If f_{E_1}, f_{E_2} be a Borel soft lattice. Then $f_{E_1 - E_2}$ is a Borel soft lattice.
Proof:
Let f_{E_2} be a Borel soft lattice then $f_{E_2^c}$ be Borel soft lattice (βsoft lattice σ-algebra).
Let $f_{E_1}, f_{E_2^c}$ be a Borel soft lattice.$\mathrm{f}_{E_1 \curlywedge E_2^c}$ is also Borel soft lattice So $\mathrm{f}_{E_1 - E_2}$ is a Borelsoft lattice.

Theorem 3.9
The union of two product Borelsoft lattice is a Borelsoft lattice.
Proof:
Let $f_{A_1 \times B_1}$, $f_{A_2 \times B_2}$ be two product Borel soft lattice βsuch that
$f_{A_1 \times B_1} \prec \beta$,$f_{A_2 \times B_2} \prec \beta$ and$f_{A_1 \times B_1} = \bigvee_{i=1}^{\infty} (f_{a_i}, f_{b_i})$,$f_{A_2 \times B_2} = \bigvee_{i=1}^{\infty} (f_{c_i}, f_{d_i})$. Then

$$\left(f_{A_1 \times B_1}\right) \curlyvee \left(f_{A_2 \times B_2}\right) = [\bigvee_{i=1}^{\infty} (f_{a_i}, f_{b_i})] \curlyvee [\bigvee_{i=1}^{\infty} (f_{c_i}, f_{d_i})]$$

$= \bigvee_{i=1}^{\infty} [(f_{a_i}, f_{b_i}) \curlyvee (f_{c_i}, f_{d_i})] \prec \beta.$
In the other word,
Let f_{A_1}, f_{A_2} be a Borel soft lattice β then $f_{A_1 - A_2}$ is also Borel soft lattice β (by Corollary 3.8).
Therefore $f_{A_1 \curlywedge A_2}$ is Borel soft lattice (βsoft lattice σ-algebra). It follows that$f_{A_1 \curlyvee A_2}$ is Borel soft lattice (βsoft lattice σ-algebra).
Let f_{B_1}, f_{B_2} be a Borel soft lattice then $f_{B_1 - B_2}$ is also Borel soft lattice(by Corollary3.8).Therefore $f_{B_1 \curlywedge B_2}$ is Borelsoft lattice it follows that$f_{B_1 \curlyvee B_2}$ is Borel soft lattice.
Now,

$$f_{A_1 \times B_1}) \curlyvee \left(f_{A_2 \times B_2}\right) = \left(f_{A_1 \curlyvee A_2}\right) \times \left(f_{B_1 \curlyvee B_2}\right)$$

Since $f_{A_1 \curlyvee A_2}$and$f_{B_1 \curlyvee B_2}$ is a Borel soft lattice and since product Borelsoft lattice is also Borelsoft lattice (by Theorem 3.3). Then $(f_{A_1 \times B_1}) \curlyvee \left(f_{A_2 \times B_2}\right)$ is Borel soft lattice.

Theorem 3.10
The intersection of two product Borelsoft lattice is a Borelsoft lattice.
Proof:
Let $f_{A_1\times B_1}$, $f_{A_2\times B_2}$ be two product Borelsoft lattice β such that
$f_{A_1\times B_1} \prec \beta$, $f_{A_2\times B_2} \prec \beta$ and $f_{A_1\times B_1} = \bigvee_{i=1}^{\infty}(f_{a_i}, f_{b_i})$, $f_{A_2\times B_2} = \bigvee_{i=1}^{\infty}(f_{c_i}, f_{d_i})$. Then

$$\left(f_{A_1\times B_1}\right) \wedge \left(f_{A_2\times B_2}\right) = [\bigvee_{i=1}^{\infty}(f_{a_i}, f_{b_i})] \wedge [\bigvee_{i=1}^{\infty}(f_{c_i}, f_{d_i})]$$
$$= \bigvee_{i=1}^{\infty}[(f_{a_i}, f_{b_i}) \wedge (f_{c_i}, f_{d_i})] \prec \beta$$

In the other word,
Let f_{A_1}, f_{A_2} be a Borel soft lattice then $f_{A_1-A_2}$ is also Borel soft lattice (by Corollary 3.8).Therefore $f_{A_1\wedge A_2}$ is Borel soft lattice β (βsoft lattice σ-algebra).
Let f_{B_1}, f_{B_2} be a Borel soft lattice then $f_{B_1-B_2}$ is also Borel soft lattice (by Corollary 3.8).Therefore $f_{B_1\wedge B_2}$ is Borel soft lattice β (βsoft lattice σ-algebra).
Now,

$$(f_{A_1\times B_1}) \wedge \left(f_{A_2\times B_2}\right) = \left(f_{A_1\wedge A_2}\right) \times \left(f_{B_1\wedge B_2}\right)$$

Since $f_{A_1\wedge A_2}$,$f_{B_1\wedge B_2}$ is a Borel soft lattice and since product Borelsoft lattice is also Borelsoft lattice (by theorem 3.3.). Then $(f_{A_1\times B_1}) \wedge \left(f_{A_2\times B_2}\right)$ is Borel soft lattice.

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