Newton-Kantorovich Method for Solving Some Types Of Nonlinear P.D.E.’S

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INTRODUCTION

The boundary value problem consists of an equation (linear or nonlinear), this equation may be (differential equation, delay differential equation) with boundary conditions (linear or nonlinear). Because of this variety, a considerable amount of numerical study has been devoted to this type of problems, [5].

This work is devoted to study the nonlinear second order ordinary differential equations which can be solved numerically by the so-called Newton-Kantorovich Method for Solving Some Types Of Nonlinear P.D.E.’S

Newton-Kantorovich method (N.K.) and generalized this method to solve some nonlinear second order partial differential equation. The Newton-Kantorovich approach replaces a nonlinear boundary problem by a sequence of linear boundary value problems, which in general converges rapidly to the true solution of the original nonlinear equation.

Consequently, a system of linear algebraic equations will be obtained and solved by some method such as MATLAB technique which becomes the tool
of currently used by engineers and applied mathematicians, so the users have an easier and more productive time in this MATLAB technique, [6].

NEWTON-KANTOROVICH METHOD
Among the techniques currently available, the Newton-Kantorovich method (N.K.) is worthwhile. The (N.K.) technique was developed in 1965 by Bellman and Kalaba [2], in connection with the dynamic programming approach. They concluded that, it is not necessary to employ the dynamic programming approach because the generalized Newton-Raphson method for operator equation yields the same results. As cited by Kubiček and Hlavaček [5], Kantorovich, McGill and Kenneth studied, in 1963, the convergence properties of the generalized Newton-Raphson method, and gave an algorithm for the numerical solution of nonlinear ordinary differential equation. The Newton – Kantorovich method was presented in two ways the first one is from Frechet derivative, and the second is from the Taylor series expansion.

1 Newton - Kantorovich method from Frechet derivative [5]
Invaluable view was introduced to explain the technique of (N.K.) and its implementation. The (N.K) method is the Newton method for an operator equation:

\[ F(y) = 0 \] ................................................................. (2.1)

The Derivation of (N.K.) method for solving the single non-linear second order differential equation is as follow:

\[ F(y) = y'' + f(x, y, y') = 0, x \in (a, b) \] ................................................. (2.2)

Subjected to the linear homogeneous two-point boundary conditions

\[ \alpha_0 y(a) + \beta_0 y'(a) = 0 \] .......................................................... (2.3)
\[ \alpha_1 y(b) + \beta_1 y'(b) = 0 \]

Where \( \alpha_0, \beta_0, \alpha_1 \) and \( \beta_1 \) are constants.

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The function f is supposed to be continuously differentiable with respect to both y and y'. Here F is the mapping from Y

\[ Y = \{y \mid y \text{satisfies equations (2.2) and (2.3)}\} \]

Into the Banach space of continuous functions \( C_{[a, b]} \).

The main formula of Newton method on the operator equation (2.1), is given by the following:

\[ F'(y_k) \delta y_k = -(y_k) .......................................................... (2.4) \]

where

\[ \delta y_k = y_{k+1} - y_k \]
Here \( F'(y) \) is the Frechet derivative of F at y [3].

\[ F'(y)\delta y = (\delta y)'' + f \]
\[ y \]
\[ \partial \]
\( \partial' (\delta y)' + f \)
\( \partial \)
\( \partial \)
\( \delta y \) ................................. (2.5)

Now, equation (2.4) could be rewritten as:

\( (\delta y_k)' + f(x, y_k, (y_k)') \)
\( \partial' \)
\( \partial \)
\( (\delta y_k)' + f(x, y_k, (y_k)' ) \)
\( \partial \)
\( \partial \)
\( \delta y_k = -(y_k)' - f(x, y_k, (y_k)') \) .......................... (2.6)

and the form of the boundary conditions will change after the following substitutions:
since \( \delta y_k = y_{k+1} - y_k \) and by using equation (2.3), one can get:
\( \alpha_0 \delta y_k(a) + \beta_0 (\delta y_k)'(a) = 0 \), \( \alpha_1 \delta y_k(b) + \beta_1 (\delta y_k)'(b) = 0 \) ........................ (2.7)

\( \alpha_0 y_{k+1}(a) + \beta_0 (y_{k+1})'(a) = 0 \), \( \alpha_1 y_{k+1}(b) + \beta_1 (y_{k+1})'(b) = 0 \)

Hence the linearized equation together with the boundary conditions take the following form:

\( (\delta y_k)' + f \)
\( \partial \)
\( \partial \)
\( (\delta y_k)' + f \)
\( \partial \)
\( \partial \)
\( \delta y_k = -(y_k)' - f(x, y_k, (y_k)') \) .......................... (2.8)

Thus; the following algorithm involved the iterative solution of the method is as follow:

1- \( k=0 \)
2- for a given an initial solution \( y_k \)
3- solve equation (2-7) and (2-8) to get \( \delta y_k \)
4- set \( y_{k+1} = y_k + \delta y_k \)
5-check the convergence :

If \( \| y_{k+1} - y_k \| < \varepsilon \), Then stop

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otherwise $k = k + 1$

6- repeat steps 3-5 until the convergence sustained.

2 Newton – Kantorovich Method from Taylor series:
The Newton-Kantorovich (N.K.) is derived by applying the Taylor series expansion. After expanding the function $f(x, y, y')$ by Taylor series expansion [5], up through first-order terms around the solution $y_k$ one can get:

$$f(x, y_{k+1}, (y_k + 1)') = f(x, y_k, (y_k)') + f'(x, y_k, (y_k))$$

and substituting the above expression in the equation (2.2) to get:

$$f(x, y_k, (y_k)') + f''(x, y_k, (y_k)) + \delta = -\partial' \partial' + \delta + \delta = -\partial' \partial'$$

The above reduced equation is identical to the equation (2.8), hence the both methods of derivation gave the same linear differential equation.

In the case of differential equation subjected to nonlinear boundary conditions, one can use the same procedure of the Taylor series expansion for the nonlinear boundary conditions, that is if the given boundary conditions are:

\[ g[y(b), y'(b)] = 0 \]
\[ g[y(a), y'(a)] = 0 \]

Then an expansion of the left-hand side of (2.10) in a Taylor series expansion up through first order terms around the solution $y_k$ is reduced to:
\[ g[y(a),(y)(a)][y(a)] \\
\quad y \partial' \delta \\
\quad \partial + \frac{0}{k}k \\
g[y(a),(y)(a)][(y)(a)] \\
\quad y \partial' \delta' \\
\quad \partial' = -g[y(a),(y)(a)]_{0k'}; \text{ and} \\
\frac{1}{k}k \\
g[y(b),(y)(b)][y(b)] \\
\quad y \partial' \delta' \\
\quad \partial' = -g[y(b),(y)(b)]_{1k'} \quad \text{(2.11)} \\
\]

According to the (N.K.) theorem, the sequence \( \{y_k\} \) converges monotonously with quadratic rate of convergence to the exact solution of the nonlinear boundary value problem, [5].

**Theorem (Newton–Kantorovich Theorem)**

Consider the operator equation:

\[ F(y) = 0 \quad \text{................................................................. (2.12)} \]

and let \( F: X \rightarrow Y \) be a map, where \( X \) and \( Y \) are Banach spaces, having a Frechet derivative at each point of an open convex set \( D \) in \( X \). Let \( [F'(y_0)]^{-1} \) exist for some point \( y_0 \in D \) and:

i- \( \| [F'(y)]^{-1} \| \leq \beta \).

ii- \( \| [F'(y)]^{-1}F(y_0) \| \leq \eta \).

iii- \( \| F'(y_1) - F'(y_2) \| \leq k\|y_1 - y_2\|; y_1, y_2 \in D \).

where \( \beta, \eta \) and \( k \) are constants which satisfy:

\[ h = \beta k \eta < 1 \]

3 Convergence of The Newton–Kantorovich Method

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iii- \( \| F'(y_1) - F'(y_2) \| \leq k\|y_1 - y_2\|; y_1, y_2 \in D \).

where \( \beta, \eta \) and \( k \) are constants which satisfy:

\[ h = \beta k \eta < 1 \]
; if
\[ t^* = \eta \]

\[ h \]

\[ [1 \ 1 \ 2h] : \text{and} \]

\[ D_{t^*} = \{ y : \| y - y_0 \| \leq t^* \} \]

Then the successive approximations given by (N.K.) method \( y_1, y_2, \ldots \), are defined for all \( n, y_n \in D_{t^*} \), and converged to a point \( y^* \), which satisfies \( F(y^*) = 0 \).

In addition:
\[ \| y_n - y^* \| \leq h \]

\[ \eta \]

\[ n \]

\[ 2 \]

\[ [1 \ 1 \ 2h] , - - - \]

, \( n = 0, 1, \ldots \)

**Proof:** See Reference [7]

3. Generalization of Newton–Kantorovich Method to Solve the Nonlinear Partial Differential Equations

In this article, the (N.K.) method will be applied to solve the nonlinear second order partial differential equations of the form:

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\[ 0 = F(u) = u_{xx} + f(x, t, u, u_x, u_t) \] ................................................... (3.1)

with initial condition of the form:

\[ u(x, 0) = g(x), a \leq x \leq b \] ................................................... (3.2)

and boundary conditions of the form:

\[ u(b, t) = f_1(t) \]

\[ u(a, t) = f_2(t) \]

\[ 0 < t < \infty \] ................................................... (3.3)

where \( g(x), f_1(t) \) and \( f_2(t) \) are continuous functions, also the function \( f \) is supposed to be continuously differentiable with respect to \( u, u_x, u_t \) and \( F \) and the \( F \) is the mapping defined from space \( U \), where:

\[ U = \{ u | u \text{ satisfies the expressions (3.1), (3.2) and (3.3)} \} \], into Banach space of continuous functions \( C([a, b]\times[0, \infty)) \).

3.1 Newton–Kantorovich Method from Taylor series Expansion:
The Newton-Kantorovich method is based on expanding the function \( f(x, t, u, u_x, u_t) \) in Taylor series expansion up through first order terms
about the $k$th iteration of the solution $u_k(x, t)$, that is:

$$
\begin{align*}
&k \frac{1}{k} x \frac{1}{k} t \\
&k \frac{x}{k} t \\
&k \frac{1}{k} k \\
&k \frac{x}{k} t \\
&k \frac{1}{k} x \\
&x \\
&\partial (x, t, u, (u_x), (u_t)) f (x, t, u, (u_x), (u_t)) \\
&f (x, t, u, (u_x), (u_t)) (u_x) (u_t) \\
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\end{align*}
$$

where $u_{k+1}$ is the $(k+1)$th iteration of the solution $u$, $(u_k)_x = u_k(x, t)$

\[ x \]

, and

$(u_k)_t = u_k(x, t)$

\[ t \]

, for $k = 0, 1, 2$......

Then equation (3.1) can be rewritten as:

$(u_{k+1})_{xx} + f(x, t, u_{k+1}, (u_{k+1})_x, (u_{k+1})_t) = 0$ ........................................... (3.5)

where $(u_{k+1})_{xx} = \frac{2}{k+1}$
By substituting eq.(3.4) in the eq.(3.5), one can get:

\[
\begin{align*}
\frac{\partial}{\partial x} (u_k) + \frac{\partial}{\partial t} (u_k) = f(x, t, u_k, (u_k)_x, (u_k)_t) + u_k + \delta u_k \\
\frac{\partial}{\partial x} (\delta u_k) = -\frac{\partial}{\partial t} (u_k) - f(x, t, u_k, (u_k)_x, (u_k)_t) \\
\end{align*}
\]

Let \( \delta u_k = u_{k+1} - u_k \), \( (\delta u)_x = (u_{k+1})_x - (u_k)_x \), \( (\delta u)_t = (u_{k+1})_t - (u_k)_t \), and \( (\delta u)_{xx} = (u_{k+1})_{xx} - (u_k)_{xx} \), then equation (3.6) was reduced to the form:

\[
\begin{align*}
(\delta u_{k+1})_{xx} + f(x, t, u_k, (u_k)_x, (u_k)_t) \\
\frac{\partial}{\partial x} (\delta u_k) + f(x, t, u_k, (u_k)_x, (u_k)_t) \\
(\delta u_k)_x + f(x, t, u_k, (u_k)_x, (u_k)_t) \\
(\delta u_k)_t = -((u_k)_{xx} - f(x, t, u_k, (u_k)_x, (u_k)_t)) \\
\end{align*}
\]

with the initial boundary conditions:

\( \delta u_k(x, 0) = g(x), a \leq x \leq b \).
The algorithm involved the iterative solution of the method is as follow:
1 - \( k=0 \)
2 - for a given an initial solution \( u_k(x, t) \)
3 - solve equation (3.7) and (3.8) to get \( \delta u_k \):
4 - set \( u_{k+1} = u_k + \delta u_k \)
5 - check the convergence:
\[
\| u_{k+1} - u_k \| < \varepsilon, \text{ Then stop}
\]
otherwise \( k=k+1 \)
6 - repeat steps 3-5 until the convergence sustained.

Remark (1):
If the given boundary conditions take the form:
\[
\alpha_0 u(a, t) + \beta_0 u_x(a, t) = 0 \quad \text{(3.9a)}
\]
\[
\alpha_1 u(b, t) + \beta_1 u_x(b, t) = 0 \quad \text{(3.9b)}
\]
The use of the Newton-Kantorovich method with \( \delta u_k = u_{k+1} - u_k \), will reduce
the above equations to the following form:
\[
\alpha_0 \delta u_k(a, t) + \beta_0 (\delta u_k)_x(a, t) = 0, \quad \alpha_1 \delta u_k(b, t) + \beta_1 (\delta u_k)_x(b, t) = 0. \quad \text{(3.10a)}
\]
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\[
\alpha_0 u_{k+1}(a, t) + \beta_0 (u_{k+1})_x(a, t) = 0, \quad \alpha_1 u_{k+1}(b, t) + \beta_1 (u_{k+1})_x(b, t) = 0. \quad \text{(3.10b)}
\]

Remark (2):
If the initial-boundary conditions are given in the following form:
\[
g_1(u(x, 0), u_t(x, 0)) = 0 \quad \text{(3.11)}
\]
\[
g(u(b, t), u(b, t)) = 0 \quad \text{(3.12)}
\]
Then, to solve the differential equation (3.1) together with the conditions (3.11) and (3.12), the left hand side of the equations (3.11) and (3.12) was expanded around the solution \( u_k(x, t) \) by Taylor series expansion up to first-order terms, that is:
\[
g_1(u_{k+1}(x, 0), (u_{k+1}(x, 0)_t) = g_1(u_k(x, 0), (u_k)_t(x, 0) +
\]
\[
g_1(u_k(x,0), (u_k)_t(x,0)) \quad \partial \quad \partial \\
(u_{k+1} - u_k + \frac{1}{k} \sum_k t) \quad g(u(x,0), (u)_t(x,0)) \quad \partial \quad \partial \\
(u_{k+1} - u_k + \frac{1}{k} \sum_k t) \quad g((u_{k+1})_t - (u_k)_t) \quad ..........(3.13) \\
g_1(u_k(x,0), (u_k)_t(x,0)) \quad \partial \quad \partial \\
(u_{k+1} - u_k + \frac{1}{k} \sum_k u \quad g((u_{k+1})_t - (u_k)_t) \quad = -g_1(u_k(x,0), (u_k)_t(x,0)) \quad ..........(3.14) \\
g_2(u_k(a,t), (u_k)_x(a,t)) \quad \partial \quad \partial \\
(u_{k+1} - u_k + 2 \frac{1}{k} \sum_k x \quad g(u(a,t), (u)_x(a,t)) \quad \partial \quad \partial \\
((u_{k+1})_t - (u_k)_t) = -g_2(u_k(a,t), (u_k)_x(a,t)) \quad and \quad \\
g_3(u_k(b,t), (u_k)_x(b,t)) \quad \partial \quad \partial \\
(u_{k+1} - u_k + 3 \frac{1}{k} \sum_k x \quad g(u(b,t), (u)_x(b,t)) \quad \partial \
\]
\[
\left(\left( u_{k+1}\right)_t \right.
- \left. (u_k)_x \right) = -g_3(u_k(b, t), (u_k)_x(b, t)) \] ...................................(3.15)

Further, equations (3.14) and (3.15) can be rewritten in the following forms:

\[
g_1(u_k(x,0), (u_k)_t(x,0))
\]

\[
\partial_t \partial u_k + \partial x_k \partial t_k 
\]

\[
g_2(u_k(a, t), (u_k)_x(a, t))
\]

\[
\partial_x \partial u_k + \partial x_k \partial x_k 
\]

\[
(\partial u_k)_x = -g_2(u_k(x, 0), (u_k)_t(x, 0)) \]........................................... (3.16)

\[
g_3(u_k(b, t), (u_k)_x(b, t)) \]

\[
\partial x \partial u_k + \partial x_k \partial x_k 
\]

\[
(\partial u_k)_x = -g_3(u_k(b, t), (u_k)_x(b, t)) \] ;and ,

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\[
g_2(u_k(a, t), (u_k)_x(a, t))
\]

\[
(\partial u_k)_x
\]
3.2 The Application of Newton-Kantorovich Method For Burger's Equation

Consider the time-dependent boundary value problem of Burger's equation, given by:
\[ M(u + u \cdot u_x) = u_{xx}, \quad 0 < x < 1, \quad t > 0 \] ........................................... (3.18)

with the boundary conditions:
\[
\begin{align*}
&u(1, t) = 0 \\
&u(0, t) = 0 \\
&t > 0
\end{align*}
\] ................................................................... (3.19)

and the initial condition:
\[ u(x, 0) = \sin \pi x, \quad 0 < x < 1 \] ....................................................... (3.20)

where \( u(x, t) \) denotes the velocity, which is a function of space and time (\( x \) and \( t \)), and \( M \) denotes a parameter, which corresponds to the Reynolds number in viscous fluid flow problems, [9]. The burgers’ equation represents a simplest integrable nonlinear equation combining the wave propagation part and diffusive effects part of the unsteady flow problem in open conveyance.

Making use \( M = 10 \), the equation (3.18) becomes:
\[ u_{xx} - 10 u_t - 10 u \cdot u_x = 0 \]
The use of equation (3.7) in the above expression yields:
\[ (\delta u_k)_{xx} - 10(\delta u_k)_x \delta u_k - 10u_k(\delta u_k)_x - 10(\delta u_k)_t = -(\delta u_k)_{xx} + 10(\delta u_k)_t + 10 \]
\[ u_k(u_k)_x \] ................................................................. (3.21)

Setting \( k = 0 \) and \( u_0(x, t) = 0 \), and for simplicity one can write \( (\delta u_0)_{xx} = \delta_{xx} \) and \( (\delta u_0)_t = \delta_t \), the following equation can be obtained:
\[ \delta_{xx} - 10 \delta_t = 0 \] ................................................................... (3.22)

The finite difference should be implemented herein to accomplish the solution. Many schemes of finite difference are available, but one of these Newton-Kantorovich Method for Solving Some Types Of Nonlinear P.D.E.'s
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schemes is tested, that is the central difference scheme which is preferable over forward and backward difference scheme. The central difference used in the (N.K.) formulation to solve the ordinary differential equation gave good results compared with exact solution, [1].

Thereby, the equivalent central difference scheme in two dimensional problem is the Crank – Nicholson implicit method, [8]. Thus, the following scheme was adopted as:

\[ 0 \]
\[ L \]
\[ 10 \]
Where $h = 0.1$ and $L = 0.01$, and after some manipulations the above expression becomes:

$$\delta_{i+1,j} + \delta_{i-1,j} + \delta_{i+1,j+1} + \delta_{i-1,j+1} + 18 \delta_{i,j} - 22 \delta_{i,j+1} = 0$$

Equation (3.23)

The initial condition of the problem takes the form:

$$u_{0,0} = \sin \pi x = \sin 0 = 0$$
$$u_{0,1} = \sin \pi (0.1) = 0.3090, u_{1,0} = 0.58778, u_{3,0} = 0.8090, u_{4,0} = 0.9510, u_{7,0} = 0.8090, u_{8,0} = 0.5877, u_{9,0} = 0.3090, u_{10,0} = 0$$

Moreover, $u_{0,1} = u_{0,2} = u_{0,3} = u_{0,4} = u_{0,5} = u_{0,6} = u_{0,7} = u_{0,8} = u_{0,9} = u_{0,10} = 0$.

Discretizing over all $i = 1, \ldots, 9$, $j = 0$, starting with $i = 1$, $j = 0$, the following expressions were obtained:

$$\delta_{2,0} + \delta_{0,0} + \delta_{2,1} + \delta_{0,1} + 18 \delta_{1,0} - 22 \delta_{1,1} = 0$$

Equation (3.24)

By the same above procedure, taking into account $i = 2, \ldots, 9$:

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$..........................(3.24)$

22 5.5469
22 4.4041
22 2.8263
22 0.789
22 0.975
22 2.82622
22 4.40274
22 5.54826
8.1 9.1
7.1 8.1 9.1
6.1 7.1 8.1
5.1 6.1 7.1
4.1 5.1 6.1
3.1 4.1 5.1
The resulting set of linear algebraic equations can be easily solved by direct method after transforming this system into the matrix form as:
5.54826
4.40274
2.82622
0.9750
0.7890
2.8263
4.4041
5.5469

The matrix solution is:
\[ \delta_{1,1} = 0.2780, \delta_{2,1} = 0.2748, \delta_{3,1} = 0.2190, \delta_{4,1} = 0.1406, \]
\[ \delta_{5,1} = 0.0489, \delta_{6,1} = -0.0400, \delta_{7,1} = -0.1402, \delta_{8,1} = -0.2185, \]
\[ \delta_{9,1} = -0.2621 \]

In order to find \( u_{1,1}, u_{2,1}, \ldots, u_{9,1} \)
Set: \( \delta_{1,1} = u_{1,1} - u_{0,1} \), therefore \( 0.2780 = u_{1,1} - 0 \); \( u_{1,1} = 0.2780 \)

Using the similar computation to get:
\[ u_{2,1} = 0.5528, u_{3,1} = 0.7718, u_{4,1} = 0.9124, u_{5,1} = 0.9613, \]
\[ u_{6,1} = 0.9213, u_{7,1} = 0.7811, u_{8,1} = 0.5626, u_{9,1} = 0.3005 \]

Now, at \( j = 1 \) one can find the second row using similar manner to obtain the following system:

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.....................................................(3.25)
Also, the linear system of algebraic equations can be written in the following matrix form:
Solving this matrix system by direct method and getting the results:

$\delta_{1,2} = 0.2522, \delta_{2,2} = 0.2687, \delta_{3,2} = 0.2166, \delta_{4,2} = 0.1393,$

$\delta_{5,2} = 0.04091, \delta_{6,2} = -0.0409, \delta_{7,2} = -0.1380, \delta_{8,2} = -0.2134,$

$\delta_{9,2} = -0.2204$

Hence:
\[ \delta_{1,2} = u_{1,2} - u_{0,2} \]
\[ 0.2522 = u_{1,2} - 0 \]
\[ u_{1,2} = 0.2522 \]

Using similar computation, the following values were obtained:
\[ u_{2,2} = 0.5209, u_{3,2} = 0.7375, u_{4,2} = 0.8768, u_{5,2} = 0.9259, u_{6,2} = 0.885, u_{7,2} = 0.747, u_{8,2} = 0.533, u_{9,2} = 0.3126. \]

The above results were compared with exact solution [4], and the absolute errors were followed in Table (1). The increments of space and time were designed through the solution to be 0.1 for both (\( \Delta x = 0.1 \) and \( \Delta t = 0.1 \)). The Fig. (1) shows the graphical representation of the exact and the numerical solution by (N.K.) method in terms of space direction at \( t = 0.2 \).

### Table (1) Comparison Between The Numerical and The Exact Solution of Burger's Equation

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<th>( x )</th>
<th>( u_{0,0} )</th>
<th>( u_{0,1} )</th>
<th>( u_{0,2} )</th>
<th>( u_{0,3} )</th>
<th>( u_{0,4} )</th>
<th>( u_{0,5} )</th>
<th>( u_{0,6} )</th>
<th>( u_{0,7} )</th>
<th>( u_{0,8} )</th>
<th>( u_{0,9} )</th>
<th>( u_{0,10} )</th>
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<td>0.0</td>
<td>0.0</td>
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<td>0.2780</td>
<td>0.2522</td>
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<td>0.9259</td>
<td>0.94237</td>
<td>0.01647</td>
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<tr>
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<td>0.6</td>
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<td>0.93743</td>
<td>0.05243</td>
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<td>0.7</td>
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</tbody>
</table>

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Fig. (1) Comparison between the Numerical solution and the Exact for Burger's Equation at \( t = 0.2 \).
The obtained results by using the current method were also compared with the solution carried out by the traditional finite difference method of the problem achieved by the authors,[1]. This comparison revealed that the (N.K.) solution with aid of central difference scheme for Burger's equation is more accurate than the numerical solution for nonlinear differential equation performed by the traditional finite difference method.

CONCLUSIONS
The following conclusions can be withdrawn from the present study:
1. The Newton-Kantorovich method transforms the nonlinear partial differential equations into linear partial differential equations, as well as, transforming the nonlinear boundary conditions into linear condition which is finally produces a linear differential equation with linear boundary conditions that could be solved using any suitable method.
2. The use of central difference scheme for solving the linear ordinary differential equation which obtained by using Newton-Kantorovich method as a type of the finite difference method gave good results compared with the exact solution. Thus; the central difference method is preferable over other difference schemes.
3. The MATLAB package is a suitable tool for solving the system of linear algebraic equations reduced from the use of (N.K.) method.
4. The traditional finite difference method used for solving nonlinear differential equations gave less accurate results compared with that obtained by Newton-Kantorovich method.

REFERENCES