Some Fixed Point Theorems For Generalized Contractive Self Mapping On Cone - b – Metric Space

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ABSTRACT

In this paper, we prove some fixed point theorems for self mapping satisfying generalized contractive condition in the setting of T – Orbitally complete cone –b-metric space with assumption that the cone is non – normal . our results give a generalization of some modern result .

Key Words: Fixed point, Contractive mapping, Cone-b metric space, Normal cone, Non-normal Cone.

INTRODUCTION

Metric fixed point theory is a branch of the fixed point theory which finds its primary application in functional analysis. It is a sub-branch of the functional analytic theory in which geometric conditions on the mapping and / or underlying space play a crucial role. Although it has a purely metric facet, it is also a major branch of nonlinear functional analysis with close ties to a Banach space geometry, [1]. Historically; the basic idea of the metric fixed point principle firstly appeared in explicit from Banach’s thesis 1922 [2,p.5], where it was used to establish the existence of solution to an integral equation. This principle Banach contraction mapping is remarkable in its simplicity contraction; it is perhaps the most widely applied fixed point theorem in all of analysis. This is because the contractive condition on the mapping is simple and easy to test because:

(i) IT requires only complete metric space for its setting.
(ii) IT provides a contractive algorithm (iterative method).
(iii) IT finds almost conociale applications in the theory of differential and integral equations specially the existence solution, uniqueness solution.

All these properties motivate authors to study this principle and there appeared many types of contraction mapping on metric space. Recently, Bakhtin [3] introduced b-metric space as a generalization of metric spaces. He proved the Contraction mapping principle in b-metric spaces that generalized the famous Banach Contraction principle in metric spaces. Since then, several papers have dealt with fixed point theory or the variation principle for single-valued and multi-valued
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operators in b-metric spaces (as shown in [4]and [5]). In [6] Haung and Zhang introduced Cone metric spaces as a generalization of metric spaces by replacing the set of real numbers by an ordered Banach space and they proved some fixed point theorems for contractive mappings by using the normality of a Cone in results which expanded certain results of fixed points in metric spaces, and other authors who worked in the same way like [7] and [8]. In [9], Hussain and Shah introduced Cone b-metric spaces as a generalization of b-metric spaces and Cone metric spaces and they established some topological properties in such space and improved some recent results about KKM mappings in the setting of a Cone b-metric space, as well as in [10] they generalized the results of [9] and obtained some fixed point theorems of contractive mappings without the assumption of normality of the Cone. In this paper, we generalized the results of [9] and [10] and prove some fixed point theorems for self mapping satisfying generalized contractive condition by using a certain vector valued altering function satisfying some properties in the setting of T-orbitally Complete cone b-metric space where the normality of the cone is omitted, we shall call this altering function by cone b-altering function.

PRELIMINARIES
Consistent with Haung and Zhang [6], the following definitions:
Let E be a normed space and P be a subset of E, P is called a Cone if:
(i) P is closed, non empty and P ≠ {0}.
(ii) ax + by ∈ P for all x, y ∈ P and non-negative real numbers a, b.
(iii) P ∩ (−P) = {0}.

Given a Cone P ⊆ E, we define a partial ordering “≤” with respect to P by x ≤ y if and only if y − x ∈ P, we write x < y to indicate that x ≤ y but x ≠ y, while x ⪯ y stand for y − x ∈ int(P), where int(P) is the interior of P.

The Cone P is called normal if there is a number k > 0 such that for all x, y ∈ E, 0 ≤ x ≤ y implies ∥x∥ ≤ k∥y∥, the least positive number satisfying the above inequality is called the normal constant of P.

Example (1): [7]
Let E = C([0,1]) with supremum norm and P ={ f ∈ E :f ≥ 0 }where ∥ f ∥ = sup {∥f(x_i)∥ , x_i ∈[0,1] } for all f, g ∈ P, put f(x) =x , g(x) =2x, then 0≤f ≤ g , ∥f∥ =1 , ∥g∥ =2 . So ∥f∥ ≤∥g∥ and K=1 . Therefore P is normal cone with normal constant K=1.

Remark (1):[7]
There are cones are not normal, the following example show that:

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Example (2):[7]
Let $E = \mathbb{C}^{R^2}(0,1]$ with the norm $\|f\| = \|f\|_\infty + \|f\|_\infty$ and consider the cone $P = \{ f \in E : f \geq 0 \}$, where $\|f\|_\infty = \max \{ |f(x_1)|, |f(x_2)|, \ldots, |f(x_n)|, x_i \in [0,1] \forall i = 1,2,\ldots,n \}$
$\|f\|_\infty = \max \{ |f'(x_1)|, |f'(x_2)|, \ldots, |f'(x_n)|, x_i \in [0,1] \forall i = 1,2,\ldots,n \}$
For each $k \geq 1$, put $f(x) = x$ and $g(x) = x^{2k}$. Then $0 \leq g \leq f$, $\|f\| = 2$ and $\|g\| = 2k + 1$, since $k \|f\| < \|g\|$, $k$ is not a normal constant of $P$. Therefore, $P$ is non-normal cone.
In the following we always suppose that $E$ is a normed space, $P$ is a cone in $E$ with $\text{int}(P) \neq \emptyset$ and $\leq$ is a partial ordering with respect to $P$.

Definition (1):[6]
Let $X$ be non-empty set, a mapping $d: X \times X \rightarrow E$ is called a Cone metric space on $X$ if the following conditions are satisfied:
(i) $0 \leq d(x,y)$ for all $x,y \in X$ with $x \neq y$ and $d(x,y) = 0$ if and only if $x = y$.
(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$.
(iii) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$.
Then the ordered pair $(X,d)$ is called a Cone metric space.

Example (3):[6]
Let $E = \mathbb{R}^2$ with usual norm on $\mathbb{R}^2$ defined by $\|x\| = \max \{ |x_1|, |x_2| \}$ for all $x \in \mathbb{R}^2$, $x = (x_1,x_2), x_i \in \mathbb{R}, i = 1,2$, $P = \{ (x,y) \in E : x,y \geq 0 \} \subset \mathbb{R}^2$, $X = \mathbb{R}$ and $d: X \times X \rightarrow E$ such that $d(x,y) = (|x-y|, \alpha |x-y|)$, where $\alpha \geq 0$ is a constant. Then $(X,d)$ is a cone metric space.

Definition (2):[9]
Let $X$ be a non-empty set and $S \geq 1$ be a given real number. A mapping $d: X \times X \rightarrow E$ is said to be Cone b-metric if and only if, for all $x,y,z \in X$, the following conditions are satisfied:
(i) $0 < d(x,y)$ with $x \neq y$ and $d(x,y) = 0$ if and only if $x = y$.
(ii) $d(x,y) = d(y,x)$.
(iii) $d(x,y) \leq S[d(x,z) + d(z,y)]$.
The pair $(X,d)$ is called a Cone b-metric space.

Example (4):[10]
Let $X = \{ 1,2,3,4 \}, E = \mathbb{R}^2$, $P = \{ (x,y) \in E : x \geq 0, y \geq 0 \}$. Define $d: X \times X \rightarrow E$ by
$$d(x,y) = \begin{cases} (|x-y|, |x-y|) & \text{if } x \neq y \\ \theta & \text{if } x = y \end{cases}$$
Then $(X,d)$ is a cone b-metric space with the coefficient $S = \frac{6}{5}$.

Definition (3):[9]
Let $(X,d)$ be a cone b-metric space, $x \in X$ and $\{ x_n \}$ be a sequence in $X$. Then
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(i) \( \{x_n\} \) converge to \( x \) whenever for every \( c \in E \) with \( 0 \ll c \), there is a natural number \( N \) such that \( d(x_n, x) \ll c \) for all \( n > N \). We denote this by \( \lim_{n \to \infty} x_n = x \) or \( x_n \to x \) (\( n \to \infty \)).

(ii) \( \{x_n\} \) is Cauchy sequence whenever for every \( c \in E \) with \( 0 \ll c \), there is a natural number \( N \) such that \( d(x_n, x_m) \ll c \) for all \( n, m > N \).

(iii) \( (X, d) \) is a complete cone-b-metric space if every cauchy sequence is convergent.

Recently, [11] introduce the definition of \( T- \) Orbitally complete cone metric space as follows; first let us recall the following definition:

Let \( T: X \to X \) be a mapping where \( X \) is a cone metric space. for each \( x \in X \), The orbit of \( T \) at \( x \) is \( O(x) = \{x, T(x), T(x^2), T(x^3), \ldots \} \) [11]

**Definition (4):** [11]

A cone metric space \( X \) is said to be \( T- \) Orbitally complete if every Cauchy sequence which is contained in \( O(x) \) for some \( x \in X \) converges in \( X \).

In this paper we generalize the above definition in the setting of cone –b- metric space.

It is clear that every complete cone –b- metric space is \( T- \) orbitally complete but the converse is not true in general, We can see that in the following example:

**Example (5)**[10]:

Let \( E=R^2 \), \( P=\{(x,y)\in E : x, y \geq 0 \} \subset E \), \( X=\[0,1) \) and \( d: XxX \to E \) such that \( d(x, y) = (|x-y|^p, \infty|x-y|^p) \). Where \( \infty \geq 0 \) and \( P > 1 \) are two constants .Then \( (X, d) \) is a cone –b- metric space (for details can see [10]).

Now define \( T:X \to X \) by \( T(x) = \frac{x}{3} \) for each \( x \in X \). \( X \) is not complete, but it is \( T- \) Orbitally complete since for any \( x \) in \( X \), \( T^n(x) \) converges to 0 .

**Definition (5):** [12]

If \( Y \) be any partially ordered set with relation “\( \leq \)” and \( f: Y \to Y \), we say that \( f \) is non-decreasing if, \( x, y \in Y, x \leq y \Rightarrow f(x) \leq f(y) \).

**Definition (6):** [12]

A function \( f: P \to P \) is called subadditive if for all \( x, y \in P \), \( f(x + y) \leq f(x) + f(y) \).

Seong-Hoon Cho [12] defined the \( \ll \)-increasing function by following:

A function \( F: P \to P \) is called \( \ll \)-increasing if for each \( x, y \in P \), \( x \ll y \) if and only if \( F(x) \ll F(y) \).

In the following we shall introduce Cone-b-altering function.
Definition (7):  
Let \((X,d)\) be a cone-b-metric space, let \(F:P \rightarrow P\) be a vector valued function, \(F\) is called a Cone-b-altering function if:
(i) \(F\) is non-decreasing, subadditive, \(\ll\)-increasing and surjective.
(ii) If, for \(\{t_n\} \subset P, \lim_{n \to \infty} F(t_n) = 0 \iff \lim_{n \to \infty} t_n = 0\)
(iii) \(F(\alpha^k t) = \alpha^k F(t)\) for \(\alpha \geq 1, k=1,2,\ldots\)

Example (6):
Let \(F(t) = t\) for all \(t \in P\) then \(F\) is Cone-b-altering function.

The following lemmas which are necessary through our work in this sequel are often used in Cone metric spaces in which the Cone need not be normal.

Lemma (1): [8]
Let \(P\) be a Cone and \(\{a_n\}\) be a sequence in \(E\). If \(c \in \text{int}(P)\) and \(0 \leq a_n \rightarrow 0\) (as \(n \to \infty\)), then there exists \(N\) such that for all \(n > N\), we have \(a_n \ll c\).

Lemma (2): [8]
Let \(x, y, z \in E\), if \(x \ll y\) and \(y \ll z\) then \(x \ll z\).

Lemma (3): [9]
Let \(P\) be a Cone and \(0 \leq u \ll c\) for each \(c \in \text{int}(P)\), then \(u = 0\).

Lemma (4): [13]
Let \(P\) be a Cone. If \(u \in P\) and \(u \leq ku\) for some \(0 \leq k < 1\), then \(u = 0\).

Main Results

Theorem (1):
Let \((X,d)\) be \(T\) –Orbitally cone –b-metric space with the coefficient \(S \geq 1\). Suppose the mapping \(T:X \rightarrow X\) Satisfy for all \(x,y \in X:\)
\[
F[d(Tx,Ty)] \leq a_1 F[d(x,y)] + a_2 F[d(Tx,x)] + a_3 F[d(Ty,y)] + a_4 F[d(Tx,y)] + a_5 F[d(Ty,x)] + \ldots (1.1)
\]
where this constant \(a_i \in [0,1)\) and \(a_1 + a_2 + a_3 + S(a_4 + a_5) < 1, i = 1,2,3,4,5\) and \(F\) be Cone-b-altering function., then \(T\) has a unique fixed point in \(X\), for each \(x \in X\), the iterative sequence \(\{T^n x\}\) converges to the fixed point.

Proof:
Let \(x_0 \in X\) be arbitrary point in \(X\). Let \(x_1=Tx_0\) and \(x_{n+1}=Tx_n=T^n x_0,\) for all \(n \in \mathbb{N}\).
First we will show that the Sequence \(\{x_n\}\) is a Cauchy sequence . Taking \(x=x_n, y=x_{n-1}\) in equation (1.1) we get :
\[
F[d(Tx_n,Tx_{n-1})] \leq a_1 F[d(x_n, x_{n-1})] + a_2 F[d(Tx_n, x_n)] + a_3 F[d(x_n, x_{n-1})] + \]

a_1 F[d(Tx_n; x_{n-1})] + a_3 F[d(x_{n-1}, x_n)] \\
F[d(x_{n+1}, x_n)] \leq a_1 F[d(x_n; x_{n-1})] + a_2 F[d(x_{n+1}, x_n)] + a_3 F[d(x_n; x_{n-1})] + a_4 F[d(x_{n+1}, x_n)] \\
\leq a_1 F[d(x_n; x_{n-1})] + a_2 F[d(x_{n+1}, x_n)] + a_3 F[d(x_n; x_{n-1})] + \sum_{i=1}^{n} F[d(x_1, x_0)] \\
\Rightarrow (1-a_2-Sa_4) F[d(x_{n+1}, x_n)] \leq (a_1 + a_3 + Sa_4) F[d(x_n, x_{n-1})] \quad \ldots (1.2) \\
Using symmetry of (1.2) in x, y we have: \\
(1-a_3-Sa_5) F[d(x_{n+1}, x_n)] \leq (a_1 + a_2 + Sa_5) F[d(x_n, x_{n-1})] \quad \ldots (1.3) \\
Now combine (1.2) and (1.3) we have: \\
F[d(x_{n+1}, x_n)] \leq \frac{2a_1 + a_2 + a_3 + S(a_4 + a_5)}{2-a_2-a_3-S(a_4+a_5)} F[d(x_n, x_{n-1})] \\
Put \lambda = \frac{2a_1 + a_2 + a_3 + S(a_4 + a_5)}{2-a_2-a_3-S(a_4+a_5)} \\
We must prove that \lambda < 1. \\
Since a_1 + a_2 + a_3 + S(a_4 + a_5) < 1 \\
\Rightarrow a_2 + a_3 + S(a_4 + a_5) < 1 - a_1 \\
\Rightarrow -a_2 - a_3 - S(a_4 + a_5) > a_1 - 1 \\
\Rightarrow 2 - a_2 - a_3 - S(a_4 + a_5) > a_1 + 1 \\
\Rightarrow 1 < \frac{2-a_2-a_3-S(a_4+a_5)}{a_1+1} \\
\Rightarrow 2a_1 + a_2 + a_3 + S(a_4 + a_5) < \frac{2a_1 + a_2 + a_3 + S(a_4 + a_5)}{a_1+1} < 1 \\
Therefore 0 \leq \lambda < 1, so we have \\
F[d(x_{n+1}, x_n)] \leq \lambda F[d(x_n, x_{n-1})] \leq \ldots \leq \lambda^n F[d(x_1, x_0)] \\
Now, for any m \geq 1, p \geq 1, it follows that \\
d(x_{m+p}, x_m) \leq S[d(x_{m+p}, x_{m+p-1}) + d(x_{m+p-1}, x_m)] \\
= Sd(x_{m+p}, x_{m+p-1}) + Sd(x_{m+p-1}, x_m) \\
\leq S [d(x_{m+p}, x_{m+p-1}) + s^2 [d(x_{m+p-1}, x_{m+p-2}) + d(x_{m+p-2}, x_m)]] \\
= Sd(x_{m+p}, x_{m+p-1}) + s^2 d(x_{m+p-1}, x_{m+p-2}) + s^3 d(x_{m+p-2}, x_m) \\
\leq Sd(x_{m+p}, x_{m+p-1}) + s^2 d(x_{m+p-1}, x_{m+p-2}) + s^3 d(x_{m+p-2}, x_m) + s^4 d(x_{m+p-3}, x_{m+p-2}) + \ldots + s^{p-1} d(x_{m+2}, x_{m+1}) + s^p d(x_{m+1}, x_m). \\
But by (i) of definition (7) of F; F is non-decreasing and sub-additive function we have: \\
F[d(x_{m+p}, x_m)] \leq F[Sd(x_{m+p}, x_{m+p-1})] + F[s^2 d(x_{m+p-1}, x_{m+p-2})] + F[s^3 d(x_{m+p-2}, x_{m+p-3})] + \ldots + F[s^{p-1} d(x_{m+2}, x_{m+1})] + F[s^p d(x_{m+1}, x_m)] \\
Also by (iii) of definition (7) we have: \\
F[d(x_{m+p}, x_m)] \leq SF[d(x_{m+p}, x_{m+p-1})] + s^2 F[d(x_{m+p-1}, x_{m+p-2})] + s^3 F[d(x_{m+p-2}, x_{m+p-3})] + \ldots + s^{p-1} F[d(x_{m+2}, x_{m+1})] + s^p F[d(x_{m+1}, x_m)] \\
\leq S^{m+p-1} F[d(x_1, x_0)] + s^2 S^{m+p-2} F[d(x_1, x_0)] + s^3 S^{m+p-3} F[d(x_1, x_0)] + \ldots + s^{p-1} S^{m+1} F[d(x_1, x_0)] + s^p F[d(x_1, x_0)].
\[ s_{\lambda}^{m+p} \left( \frac{1}{s_{\lambda}^{-1} \lambda^{p-1} - 1} \right) F[d(x_1, x_0)] + s_{p}^{m} F[d(x_1, x_0)] \]
\[ \leq \frac{s_{\lambda}^{m+1}}{s_{\lambda}^{- \lambda}} F[d(x_1, x_0)] + s_{p}^{m+1} F[d(x_1, x_0)] \rightarrow 0 \text{ as } m \rightarrow \infty \]

Hence, \( \lim_{n,m \rightarrow \infty} d(x_{m+p}, x_m) = 0 \) by (ii) of definition (7) of \( F \). So by lemma (1), there exists \( k \in N \) such that \( d(x_{m+p}, x_m) \ll c \) for each \( c \in \text{int}(P) \) and for all \( m > k. \) if \( n=m+p \), so for all \( m,n > k \), \( \{x_n\} = T^n(x_0) \) is a Cauchy sequence in \( X \). Since \( X \) is \( T \)- Orbitally complete, there exist \( Z \in X \) such that

\[ \lim_{n \rightarrow \infty} \{d(x_n, Z)\} = 0 \]

Let \( C \in \text{int}(P) \) be given .We can choose \( n_0 \in N \) such that \( d(x_{n_0}, x_{n_0-1}) \ll F^1 \)

\[ \left( \frac{1-a_2 - sa_4 c}{a_3} \right) F[d(x_n, Z)] \ll \left( \frac{1-a_2 - sa_4 c}{a_3 + 1} \right), d(x_{n-1}, z) \ll \left( \frac{1-a_2 - sa_4 c}{a_3 + 1} \right) \]

By (i) of definition (7)of \( F \) we have for all \( n>n_0 \ :

\[ F[d(x_{n}, x_{n-1})] \ll \left( \frac{1-a_2 - sa_4 c}{a_3} \right), F[d(x_n, z)] \ll \left( \frac{1-a_2 - sa_4 c}{a_3 + 1} \right) \]

Then we have :

\[ F[d(T_z, z)] \leq F[d(T_z, T_{x_{n+1}})] + F[d(T_{x_{n+1}}, z)] \]
\[ \leq a_1 F[d(z, x_{n-1})] + a_2 F[d(T_z, z)] + a_3 F[d(T_{x_{n+1}}, x_{n-1})] + a_4 F[d(T_z, T_{x_{n+1}})] + a_5 F[d(x_{n+1}, x_{n-1})] + F[d(x_n, z)] \]
\[ + Sa_4 F[d(T_z, Z)] + a_5 F[d(x_n, z)] \]
\[ (1-a_2 Sa_4) F[d(T_z, Z)] \leq (a_1 + a_4) F[d(z, x_{n-1})] + a_3 F[d(x_n, x_{n-1})] \]
\[ + (a_5 + 1) F[d(x_n, Z)] \]
\[ F[d(T_z Z)] \leq \frac{1-a_2 - sa_4}{F[d(z, x_{n-1})]} + \frac{a_3}{1-a_2 - sa_4} F[d(x_n, x_{n-1})] + \frac{a_5 + 1}{1-a_2 - sa_4} F[d(x_n, Z)] \]

Which implies by lemma (2):

\[ F[d(T_z, Z)] \ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = C \]

Thus by lemma (3) we have \( F[d(T_z, Z)] = 0 \) and so by (ii) of definition(7) of \( F \), We obtain that \( d(T_z, z) = 0 \) and so \( T_z = Z \)
For the uniqueness of fixed point of \( t \), suppose that \( u \) is another fixed point of \( T \), Then from equation (1.1) we have:

\[
F[d(Z,u)] = F[d(T_x,T_u)] \leq a_1 F[d(z,u)] + a_2 F[d(T_x,Z)] + a_3 F[d(T_u,u)] + a_4 F[d(T_z,u)] + a_5 F[d(T_u,z)]
\]

\[
\leq a_1 F[d(z,u)] + a_2 F[d(Z,Z)] + a_3 F[d(u,u)] + a_4 F[d(Z,u)] + a_5 F[d(u,z)]
\]

\[
\leq (a_1 + a_4 + a_5) F[d(Z,u)]
\]

But \( S \geq 1 \), so we have

\[
F[d(Z,u)] \leq (a_1 + a_2 + a_3 + S(a_4 + a_5)) F[d(Z,u)]
\]

Since \( a_1 + a_2 + a_3 + S(a_4 + a_5) < 1 \), so by lemma (4) we have \( F[d(Z,u)] = 0 \) and so \( Z = u \).

Therefore, \( T \) has a unique fixed point.

Now we have the following corollary.

**Corollary (2):**

Let \( (X,d) \) be \( T \)-orbitally complete Cone-b-metric space with the coefficient \( S \geq 1 \). Suppose the mappings \( T:X \rightarrow X \) satisfy for all \( x, y \in X \):

\[
d(Tx, Ty) \leq a_1 d(x,y) + a_2 d(Tx,x) + a_3 d(Ty,y) + a_4 d(Tx,y) + a_5 d(Ty,x) \quad (1.4)
\]

where the constant \( a_i \in [0,1) \) and \( a_1 + a_2 + a_3 + S(a_4 + a_5) < 1 \). Then \( T \) has a unique fixed point.

**Proof:**

By taking \( F(a) = a \) for all \( a \in P \) in equation (1.1) of theorem (1), we obtain the required result.

Now we give an application to explain our main result by considering the following example:

**Example (7):**

Let \( x = \mathbb{R} \), \( E = \mathbb{R}^2 \), \( P = \{(x,y) \in E : x \geq 0, y \geq 0\} \), \( d:X \times X \rightarrow E \) by

\[
d(x,y) = (|x-y|, \frac{1}{2}|x-y|) \quad (X,d) \) is complete cone -b metric space with \( s = 2 \)
\]

(for the detail is see[10] )

So \( (x,d) \) is \( T \)-orbitally complete cone -b-metric space.

Define \( T:X \rightarrow X \) by \( T(X) = \frac{1}{2} x - \frac{1}{4} x^2 \) for all \( x \in X \).

Define \( F:P \rightarrow P \) by \( F(t) = t \), for all \( t \in P \).

Therefore ; \( F[d(T_x,T_y)] = d(T_x,T_y) \)

\[
= (|T_x-T_y|, \frac{1}{2}|T_x-T_y|)
\]
\[(\frac{1}{2}x - \frac{1}{4}x^2 - \frac{1}{2}y + \frac{1}{4}y^2) , \frac{1}{2} |\frac{1}{2}x - \frac{1}{4}x^2 - \frac{1}{2}y + \frac{1}{4}y^2|)\]

\[=\left(\frac{1}{2}(x-y) - \frac{1}{4}(x^2-y^2)\right) , \frac{1}{2} |\frac{1}{2}(x-y) - \frac{1}{4}(x^2-y^2)|\]

\[=\left(\frac{1}{2}(x-y) - \frac{1}{4}(x-y)(x+y)\right) , \frac{1}{2} |\frac{1}{2}(x-y) - \frac{1}{4}(x-y)(x+y)|\]

\[\leq\frac{1}{2} (|x-y| , \frac{1}{2} |x-y|)\]

\[=\frac{1}{2} d(x,y) = \frac{1}{2} F[d(x,y)]\]

Thus ; \(F[d(T_x, T_y)] \leq \frac{1}{2} F[d(x, y)] + \frac{1}{18} F[d(x, T_x)] + \frac{5}{18} F[d(y, T_y)] + \frac{1}{36} F[d(x, T_y)] + \frac{1}{36} F[d(y, T_x)]\]

Where \(a_1 = \frac{1}{2}\), \(a_2 = \frac{1}{18}\), \(a_3 = \frac{5}{18}\), \(a_4 = \frac{1}{36}\), \(a_5 = \frac{1}{36}\)

Such that \(a_1 + a_2 + a_3 + a_4 + a_5 = \frac{1}{2} + \frac{1}{18} + \frac{5}{18} + 2(\frac{1}{36} + \frac{1}{36})\)

\[= \frac{1}{2} + \frac{6}{18} + \frac{1}{9} = \frac{9 + 6 + 2}{18} = \frac{17}{18} < 1\]

Therefore all conditions of theorem (1) are satisfied.

**REFERENCES**


