Quasi -Fully Cancellation Modules

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Abstract
Let M be an R-module. In this paper we introduce the concept of quasi-fully cancellation modules as a generalization of fully cancellation modules. We give the basic properties, several characterizations about this concept. Also, the direct sum and the localization of quasi-fully cancellation modules are studied.

Keywords: Fully cancellation modules, Quasi-fully cancellation modules.

المقاسات شبه الحذف التامة

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الخلاصة
ليكن Mمقاسا على R.في هذا البحث قدمنا المقاسات شبه الحذف التامة كتعميم لمقاسات الحذف التامة لقد اعطينا الخواص الاساسية والعديد من التشخيصات لهذا المفهوم .كذلك الجمع المباشر والتموضع لمقاسات شبه الحذف التامة قد درست.

Introduction

Let M be an R-module, where R is a commutative ring with unity. Gilmer in [1] introduce the concept of cancellation ideal, also D.D. Anderson and D.F. Anderson in [2] studied cancellation ideals, where an ideal I of a ring R is said to be cancellation if for each ideals J,K of R ,IJ = IK implies J = K. In [3], A.S. Mijbass give generalizations of this concept namely cancellation modules (weakly cancellation module , where an R-module M is called cancellation (weakly cancellation) if whenever I and J are ideals of R , IM = JM implies I = J ($I+ann_R M = J+ ann_R M$).

In[4] ,the concept of naturally cancellation modules and fully cancellation module were given,where an R-module M is called naturally cancellation if whenever ,$N\leq M$, $W\leq M$,$K\leq M$, N.W = N.K $\Rightarrow$W = K, where N.K is define by (N:M)(K:M)M. And M is called fully cancellation if for each ideal I of R and for each submodules N,W of M, IN = IW implies N = W.

In this paper we introduce the concept of quasi-fully cancellation modules, where an R-module M is called quasi-fully cancellation module if for each ideal I of R and for each submodules N, W of M such that IN = IW implies $N+ ann_M I = W+ ann_M I$.

In \$2 of this paper we give the basic properties of these classes of modules, such as every module over a principle ideal ring is a quasi-fully cancellation (See Th.2.5).

In \$3, we give many characterizations for quasi-fully cancellation modules.

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$4 is devoted to study the direct sum of quasi-fully cancellation modules. Also we study the behavior of quasi-fully cancellation modules under localization.

Quasi-fully cancellation

In this section, we will introduce a new concept namely quasi-fully cancellation modules .We give some basic results and properties of this concept. Also, relationships between this class and other types of modules are established.

Definition 2.1: Let M be an R-module, M is called quasi-fully cancellation module if to every ideal I of R and for every two submodules A,B of M such that IA = IB implies A+ $ann_M I$ = B+ $ann_M I$ (where $ann_M I=\{m\in M, \ni Im=0\}$).

Recall that an R-module M is called fully cancellation if for every non zero ideal I of R and for every submodules N and W of M such that IN = IW then N = W [4,Def. 2.1].

Remarks and Examples 2.2

(1) Every fully cancellation module is quasi-fully cancellation module

Proof: It is clear.

(2) Z as Z-module is quasi-fully cancellation module

Proof:

By [4] Z as Z-module is fully cancellation and by (1) Z is quasi-fully cancellation. □

(3) Z_4 as Z-module is quasi-fully cancellation.

Proof:

Take I=<4> and A=$<\bar{2}>$,B= Z_4 ,then IA = IB implies <4>.$<\bar{2}>$=<4> Z_4=$<\bar{0}>$. Now A+ $ann_M I$ = $<\bar{2}>$ + ann_{Z_4} <4> = $<\bar{2}>$ + Z_4 = Z_4 , also ,B+$ann_M I$ = Z_4 + ann_{Z_4} <4> = Z_4+ Z_4 = Z_4. Similarly all other cases.However Z_4 as Z-module is not fully cancellation [4] since <4> $<\bar{2}>$= <4> Z_4 but $<\bar{2}>$ $\neq Z_4$. □

(4) Every submodule of quasi-fully cancellation is quasi-fully cancellation.

Proof:

Let H, K are two submodules of N and I be an ideal of R such that IH = IK. Then H, K are submodules of M .Since M is quasi-fully cancellation,

H + $ann_M I$ =K + $ann_M I$

But $ann_N I$ = $ann_M I \cup N$ hence

H + $ann_N I$ =H + ($ann_M I \cup N$)

= (H + $ann_M I$)$\cup$N by modular law

= (K + $ann_M I$)$\cup$N

= K +($ann_M I \cup N$) by modular law

= K +$ann_N I$. □

(5) Let M_1, M_2 are two modules. If $M_1 \approx M_2$,then M_1 is quasi-fully cancellation module if and only if M_2 is quasi-fully cancellation module .

Proof:

(⇒) Let f: $M_1 \to M_2$ be an isomorphism and suppose that IN = IW, where I is an ideal of R and N, W are two submodules of M_2 .

Now, since f is onto then there exists two submodules K_1 , K_2 of M_1 such that f(K_1) = N and f(K_2) = W, so If(K_1) = If(K_2), but if f a homorphism then f(IK_1) =f(IK_2),also since f is one to one then IK_1 = IK_2. But M_1 is quasi-fully cancellation module, hence K_1+ $ann_{M_1} I$ = K_2+ $ann_{M_1} I$. Now f(K_1+ $ann_{M_1} I$)=f(K_2+ $ann_{M_1} I$) and then f(K_1)+f($ann_{M_1} I$)=f(K_2)+f($ann_{M_1} I$). One can easily show that f($ann_{M_1} I$) = $ann_{M_2} I$, hence N+ $ann_{M_2} I$ =W+ $ann_{M_2} I$.

(⇐) Clear .□

An R-module M is called naturally cancellation if whenever N, W_1 and W_2 are submodules of M such that N.W_1 = N, W_2 then W_1 = W_2, where N.W = (N:M)(W:M)M for every submodule W of M.[4].

Also, an R-module M is called multiplication if for each N≤M, there exists an ideal I of R such that N = IM. Equivalently, M is a multiplication R-module if for each submodule N of M, N = $(N:_R M)M$, where $(N:_R M)=\{r\in R: rM \subseteq N\}$[5,6].

Now, we need the following theorem to prove the next proposition.

Theorem 2.3

If M is a multiplication R-module. Then M is naturally cancellation module if and only if M is fully cancellation module.

Proposition 2.4

Let M is a multiplication and naturally cancellation module then M is quasi-fully cancellation module.

Proof:

Since M is multiplication and naturally cancellation module, so by (Rem. and Ex. (2.2) (1)) M is fully cancellation and by (Rem. and Ex.(2.2) (1)) M is quasi-fully cancellation.□

An R-module M is called torsion free if T(M) = 0 where $T(M)=\{x\in M: nx = 0,\text{for some } n\in N\}$,[6]. We proved in (Rem. and Ex. (1)) that every fully cancellation module is quasi-fully cancellation module, now in the following proposition, the converse of this statement is true under the class of torsion free modules.

Proposition 2.5

Let M be a torsion free module over integral domain R. If M is quasi-fully cancellation module then M is fully cancellation.

Proof:

Suppose IN = IW, where N, W are two submodules of M and 0≠I is an ideal of R. But M is quasi-fully cancellation thus N + $ann_M I$ = W + $ann_M I$. Now, let $x\in ann_M I$, then Ix = 0and hence rx = 0 for every $r\in I$. But I≠0, hence x =0, since M is torsion free, thus $ann_M I = 0$.therefore N = W and hence M is fully cancellation.□

Now, we have the following theorem.

Theorem 2.6

Every module on a principle ideal ring is quasi-fully cancellation module.

Proof:

Let M be a module on a P.I.R. R and let N,W are two submodules of M. Now, suppose that IN = IW, where I⊆R, since R is PIR, thus I = <a>, where $a\in R$. Hence <a>N = <a>W, then an = aw, where $n\in N$ and $w\in W$. Therefore an-aw = 0,then a(n-w) = 0. Thus $n-w \in ann_M I$. Now,n = $w+n-w\in W+ ann_M I$. Then $N\subseteq W+ ann_M I$. Hence $N+ ann_M I\subseteq W+ ann_M I$. Similarly $W + ann_M I\subseteq N + ann_M I$. Thus $N + ann_M I = W + ann_M I$, and hence M is quasi-fully cancellation module. □

By this theorem we have every Z-module is quasi-fully cancellation. In particular the Z-module $Z_{P\infty}$ is quasi-fully cancellation. However $Z_{P\infty}$ as Z-module is not fully cancellation [4].

Mijbass [3] proved that if M is an R-module and $\overline{R}=\frac{R}{annM}$ then M is a cancellation $\overline{R}$ -module if and only if M is weak-cancellation R-module.

Now, we have the following theorem.

Theorem 2.7

M is quasi-fully cancellation R-module if and only if M is quasi-fully $\overline{R}=\frac{R}{annM}$ -module.

Proof:

(⇒)

Let I be an ideal of $\overline{R}=\frac{R}{annM}$,and N,W are two $\overline{R}$ -submodules. Then $I=\frac{I^{/}}{annM}$,for some $I^{/}$ □annM and N,W are R-submodules.

Now, suppose IN = IW and we have for any $x \in I'$, $x + annM \in I$, then $(x + annM) = xn \in I'n$, for every $n \in N$. But $(x + annM)N \in IN = IW$, where $x + annM \in I$, thus $xn \in IW$, then $xn = \sum_{i=1}^{k} \overline{a_i} w_i$, where $\overline{a_i} \in I$, $w_i \in W$. But for every i, $1 \leq i \leq k$, $\overline{a_i} = a_i + annM$ and hence $xn = \sum_{i=1}^{k} (a_i + annM) w_i = \sum_{i=1}^{k} a_i w_i \in I'W$. Therefore $I'N \subseteq I'W$. Similarly $I'W \subseteq I'N$, thus $I'N = I'W$ and since M is quasi-fully cancellation, then $N + ann_M I' = W + ann_M I'$.

We claim that $ann_M I' = ann_M I$, let $m \in ann_M I'$, then $I'm = 0$. But $(I' + annM)m = I'm = 0$, thus $m \in ann_M I$ and then $ann_M I' \subseteq ann_M I$. Now, let $m \in ann_M I$, then $Im = 0$. Since $I = \frac{I'}{annM}$, then for every $y \in I'$, $y + annM \in I$, so $(y + annM)m = 0$. Therefore $xy = 0$ and hence $m \in ann_M I'$. Thus $ann_M I \subseteq ann_M I'$ and then $ann_M I = ann_M I'$. So that $N + ann_M I = W + ann_M I$.

(⇐) The proof is similarly. □

Characterizations for quasi-fully cancellation modules

In this section we will give two characterizations for quasi-fully cancellation modules.

Compare the following theorem with (4,Th.2.6)

Theorem 3.1

Let M be an R-module. Let A, B and C are submodules of M, and I be an ideal of R. Then the following statements are equivalent:-

1- M is a quasi-fully cancellation module.
2- If $IA \subseteq IB$, then $A \subseteq B + ann_M I$.
3- $I(a) \subseteq IC$, then $a \in C + ann_M I$, where $a \in M$.
4- $(IA:_M I) = A + ann_M I$.

Proof:

(1)⇒(2)

Let M be a quasi-fully cancellation module and suppose that $IA \subseteq IB$. Then $IB = IA + IB = I(A+B)$. But M is quasi-fully cancellation, thus $B + ann_M I = (A+B) + ann_M I$. Therefore $A \subseteq B + ann_M I$.

(2)⇒(3)

Let $I(a) \subseteq IC$, by (2) we have $(a) \subseteq C + ann_M I$. Thus $a \in C + ann_M I$.

(3)⇒(4)

We want to prove that $(IA:_M I) = A + ann_M I$. Let $x \in (IA:_M I)$, hence $Ix \subseteq IA$ and hence by (3) $x \in = A + ann_M I$. Therefore $(IA:_M I) \subseteq A + ann_M I$. Now, let $y \in A + ann_M I$, then $y = a + t$, where $a \in A$ and $t \in ann_M I$, i.e $It = 0$ and hence $Iy = Ia + It$. Thus $Iy = Ia \in IA$, then $y \in (IA:_M I)$. Therefore $A + ann_M I \subseteq (IA:_M I)$, and hence $(IA:_M I) = A + ann_M I$.

(4)⇒(1)

Suppose $IA = IB$, then $B \subseteq (IA:_M I)$. By (4), $(IA:_M I) = A + ann_M I$. Hence $B \subseteq A + ann_M I$, therefore $B + ann_M I \subseteq A + ann_M I$. Similarly $A \subseteq (IB:_M I)$. By(4) $(IB:_M I) = B + ann_M I$, hence $A \subseteq B + ann_M I$. Then $A + ann_M I = B + ann_M I$, thus M is quasi-fully cancellation module. □

The following theorem is another characterization for quasi-fully cancellation module.

Theorem 3.2

Let M be an R-module. Then M is quasi-fully cancellation if and only if $((N + ann_M I):_R W) = (IN:_R IW)$, where I is an ideal of R and N,W are two submodules of M.

Proof:

(⇒) Let $x \in ((N + ann_M I):_R W)$, then $xW \subseteq N + ann_M I$ and hence $xw \in N + ann_M I$, for any $w \in W$, thus $xIW \subseteq IN$, then $x \in (IN:_R IW)$. Therefore $((N + ann_M I):_R W) \subseteq (IN:_R IW)$.

Now, let $t \in (IN:_R IW)$, then $tIW \subseteq IN$ and by Th.3.1 we get $tW \subseteq N + ann_M I$ and hence $t \in ((N + ann_M I):_R W)$. Thus $((N + ann_M I):_R W) = (IN:_R IW)$.

(⇐) Now, suppose $IW \subseteq IN$, where I is an ideal of R and N,W are two submodules of M. Then $(IN:_R IW)=R$. But by assumption $((N + ann_M I):_R W) = (IN:_R IW)$ and hence $((N + ann_M I):_R W) = R$. It follows that $W \subseteq N + ann_M I$ and then M is quasi-fully cancellation module. □

Direct sum and the localization of quasi-fully cancellation modules

In this section we study the direct sum of quasi-fully cancellation modules; also we study the localization of quasi-fully cancellation modules.

Lemma 4.1

Let $M = M_1 \oplus M_2$ be an R-module, M_1, M_2 are two submodules of M. If $I \subseteq R$ then $ann_M I = ann_{M_1} I \oplus ann_{M_2} I$

Proof:

Let $m \in ann_M I$. Hence $m \in M$ and $Im = 0$. But $m \in M$ implies $m = m_1 + m_2$, for some $m_1 \in M_1$, $m_2 \in M_2$. Hence $Im = I(m_1 + m_2) = Im_1 + Im_2 = 0$. It follows that $Im_1 = Im_2 = 0$ (since + on M is a direct sum). Thus $m_1 \in ann_{M_1} I$ and $m_2 \in ann_{M_2} I$ and then $m = m_1 + m_2 \in ann_{M_1} I \oplus ann_{M_2} I$.

Conversely, let $m \in ann_{M_1} I \oplus ann_{M_2} I$, then $m = m_1 + m_2$ where $m_1 \in ann_{M_1} I$, $m_2 \in ann_{M_2} I$. So that $Im_1 = 0$ and $Im_2 = 0$, thus $Im_1 + Im_2 = I(m_1 + m_2) = Im = 0$; that is $m \in ann_M I$. Therefore $ann_{M_1} I \oplus ann_{M_2} I \subseteq ann_M I$. □

Now, we have the following theorem

Theorem 4.2

Let $M = M_1 \oplus M_2$ be an R-module, M_1, M_2 are two submodules of M and every submodule of M is fully invariant. Then M_1 and M_2 are quasi-fully cancellation if and only if M is quasi-fully cancellation.

Proof:

(⇒) Let A, B are two submodules of M and suppose IA = IB, where I is an ideal of R. Then $A = (A \cup M_1) \oplus (A \cup M_2)$ and $B = (B \cup M_1) \oplus (B \cup M_2)$[1,Prop 4.5]. Thus $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$, where $A_i = A \cup M_i$ and $B_i = B \cup M_i, \forall i$, I = 1,2. Then $I(A_1 \oplus A_2) = I(B_1 \oplus B_2)$, hence $IA_1 \oplus IA_2 = IB_1 \oplus IB_2$. Therefore $IA_1 = IB_1$ and $IA_2 = IB_2$. But M_1 and M_2 are quasi-fully cancellation, thus $A_1 + ann_{M_1} I = B_1 + ann_{M_1} I$ and $A_2 + ann_{M_2} I = B_2 + ann_{M_2} I$. It follows that $A_1 + A_2 + ann_{M_1} I + ann_{M_2} I = B_1 + B_2 + ann_{M_1} I + ann_{M_2} I$, then we have $A + ann_M I = B + ann_M I$. Therefore M is quasi-fully cancellation.

(⇐) It follows directly by (Rem.& Ex.(2.2)(3)). □

Theorem 4.3

Let $M = M_1 \oplus M_2$ be an R-module, M_1, M_2 are two submodules of M and every submodule of M such that $ann(M_1)+ann(M_2) = R$. Then M is quasi-fully cancellation if and only if, M_1, M_2 are quasi-fully cancellation.

Proof:

(⇒) Let A and B are two submodules of M and I an ideal of R. Since $ann(M_1)+ ann(M_2)=R$, then by [1,Prop 4.2], $N = N_1 \oplus N_2$ and $W = W_1 \oplus W_2$, and then by the similar procedure as in the Th.4.2 the required result can be obtained.

(⇐) Clear by Rem.&Ex. 2.2(4)). □

Next we shall study the behavior of quasi-fully cancellation modules under localization.

First compare the following result with [4,Prop 4.4]

Theorem 4.4

Let M be a module over a Noetherian ring . If M_P is quasi-fully cancellation (for every maximal ideal P of R)R_P-module, then M is quasi-fully cancellation R-module.

Proof:

Let I be an ideal of R and let N and W be submodules of M such that IN = IW. Hence $(IN)_P = (IW)_P$ .Then by [7 Exc9.11(i),P172] $I_P . N_P = I_P . W_P$ and since M_P is a quasi-fully cancellation R_P-module N_P

$+ ann_{M_P} I_P = W_P + ann_{M_P} I_P$. But I is a finitely generated ideal, since R is Noetherian, so by [7,Exc 9.13] $ann_{M_P} I_P = (ann_M I)_P$. It follows that $N_P + (ann_M I)_P = W_P + (ann_M I)_P$ and then by [7, Exc9.11 (iii)] $(N_P + ann_M I)_P = (W_P + ann_M I)_P$. Thus by [4, Lemma 4.1] $N + ann_M I = W + ann_M I$, and so M is quasi-fully cancellation module. □

Theorem 4.5

Let M be a module over a Noetherian ring R. If M is quasi-fully cancellation R_P-module, for each maximal ideal P of R, provided that for each submodule L of M_P , $L = K_P$ for some $K \subseteq M$.

Proof:

Let J be an ideal of R_P .Then there exists an ideal I of R such that $J = I_P$. Let $L, L^{/}$ be two submodules of M_P such that $JL = JL^{/}$,By hypothesis , $L = N_P$, $L^{/} = W_P$ for some $N \leq M, W \leq M$. Thus $I_P.N_P = I_P.W_P$. Hence $(IN)_P = (IW)_P$ (See[7 Exc.9.11]) and then [4,Lemma 4.1],IN = IW. But M is quasi-fully cancellation, so $N + ann_M I = W + ann_M I$. Hence $(N + ann_M I)_P = (W + ann_M I)_P$. By [7,Exc.9.11] $N_P + (ann_M I)_P = W_P + (ann_M I)_P$. But I is a finitely generated ideal, so $(ann_M I)_P = (ann_M I)_P$ (by [7,Exc.9.13]). Thus $N_P + ann_{M_P} I_P = W_P + ann_{M_P} I_P$; that is $L + ann_{M_P} J = L^{/} + ann_{M_P} J$. Thus M_P is a quasi-fully cancellation R_P-module. □

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