Generalized mean function for n-variable

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ABSTRACT:

In this work we present the theory of an integral mean for generalized GN'-function. We will show under what conditions the mean function is a GN'-function and satisfies a $\Delta$-condition. Moreover, we examine how the minimizing points in the definition of the mean function affect a basic property of the ordinary integral mean.

Keywords : Generalized GN'-function for n-variable, $\Delta$-condition, Generalized mean function.

1. Introduction and Basic Concept:

In what follows $T$ will denote a space of point with $\sigma$-finite measure and $E^n$ a n-dimensional Euclidean space.

Definition 1.1 Orlicz (1932)

Orlicz space $L_M = L_M(\Omega, \mu)$ is a Banach space consisting of all $f \in S(\Omega, \mu)$ where $S(\Omega, \mu)$ is a ring of all measurable functions on the space with bounded measure space $(\Omega, \mu)$.

such that

$$\int_{\Omega} M(|f|) d\mu < \infty,$$

With the Luxemburg Nakano norm

$$\|f\|_M = \inf \{ \lambda > 0 : \int_{\Omega} \frac{|f|}{\lambda} d\mu \leq 1 \}$$
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Orlicz spaces $L^m$ are natural generalization of $L^p$ space, where $L^p(I)$ consists of all the measurable functions $f$ defined on the interval $I$ for which

$$\left(\int_I |f|^p\right)^{\frac{1}{p}} < \infty \quad \text{Corothers(2000)}$$

They have very rich topological and geometrical structures; they may possess peculiar properties that do not occur in an ordinary $L^p$ space.

**Definition 1.2 Borwein (1997)**

Let $M : I \rightarrow \mathbb{R}$ be defined on some interval of the real line $\mathbb{R}$. A function $M$ is called convex if

$$M\left(\frac{u_1 + u_2}{2}\right) \leq \frac{1}{2} \left(M(u_1) + M(u_2)\right) \quad (1)$$

for all $u_1, u_2 \in I$

we can generalize the inequality (1) for any $u_1, u_2, \ldots, u_n$ by

$$M\left(\frac{u_1 + u_2 + \ldots + u_n}{n}\right) \leq \frac{1}{n} \left(M(u_1) + \ldots + M(u_n)\right) \quad (2)$$

**Definition 1.3 Hassen(2007)**

Let $M(t, x, y)$ be a real valued non-negative function defined on $T \times E^n \times E^n$ such that:

(i) $M(t, x, y) = 0$ if and only if $x, y$ are the zero vectors $x, y \in E^n$, $\forall t \in T$

(ii) $M(t, x, y)$ is a continuous convex function of $x, y$ for each $t$ and a measurable function of $t$ for each $x, y$

(iii) For each $t \in T$, \[ \lim_{\|x\|, \|y\| \to \infty} \frac{M(t, x, y)}{\|x\|\|y\|} = \infty, \] and...
There are constants $d \geq 0$ and $d_i \geq 0$ such that

$$\inf_{t \in \mathbb{R}} \inf_{c \geq d} k(t, c, c') > 0$$  \hspace{1cm} (1)$$

Where

$$k(t, c, c') = \frac{M(t, c, c')}{\overline{M}(t, c, c')}$$

$$\overline{M}(t, c, c') = \sup_{x \in \mathbb{R}, |x| = c} M(t, x, y), M(t, c, c') = \inf_{x \in \mathbb{R}, |x| = c} M(t, x, y)$$

and if $d > 0$ and $d_i > 0$, then $\overline{M}(t, d, d_i)$ is an integrable function of $t$. We call the function satisfying the properties (i)-(iv) a generalized N*-function or a GN*-function.

**Definition 1.4:**

Let $M(t, x_1, x_2, \ldots, x_n)$ be a real valued non-negative function defined on $T \times E^n \times E^n \times \ldots \times E^n$ such that:

(i) $M(t, x_1, x_2, \ldots, x_n) = 0$ if and only if $x_1, x_2, \ldots, x_n$ are the zero vectors $x_1, x_2, \ldots, x_n \in E^n$, $\forall t \in T$

(ii) $M(t, x_1, x_2, \ldots, x_n)$ is a continuous convex function of $x_1, x_2, \ldots, x_n$ for each $t$ and a measurable function of $t$ for each $x_1, x_2, \ldots, x_n$,

(iii) For each $t \in T$, $\lim_{\|x\| \to \infty} \frac{M(t, x_1, x_2, \ldots, x_n)}{\|x_1\| \|x_2\| \ldots \|x_n\|} = \infty$, and

(iv) There are constants $d_1 \geq 0, d_2 \geq 0, \ldots, d_n \geq 0$ such that

$$\inf_{t \in \mathbb{R}} \inf_{c \geq d_1} \inf_{c' \geq d_2} k(t, c, c', \ldots, c_n) > 0$$  \hspace{1cm} (1)$$
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Where

\[ k(t, c_1, c_2, \ldots, c_n) = \frac{M(t, c_1, c_2, \ldots, c_n)}{M(t, c_1, c_2, \ldots, c_n)} \]

\[ \overline{M}(t, x_1, c_2, \ldots, c_n) = \sup_{\|s\|_1 = 1} M(t, x_1, x_2, \ldots, x_n), \]

\[ \underline{M}(t, c_1, c_2, \ldots, c_n) = \inf_{\|s\|_2 = c_2} M(t, x_1, x_2, \ldots, x_n) \]

and if \( d_1 > 0, d_2 > 0, \ldots, d_n > 0 \), then \( \overline{M}(t, d_1, d_2, \ldots, d_n) \) is an integrable function of \( t \). We call the function satisfying the properties (i)-(iv) a generalized N'-function or a GN'-function.

**Definition 1.5 Hassen (2010)**

We say that a GN* -function \( M(t,x,y) \) satisfies a \( \Delta - \) condition if there exists a constant \( K \geq 2 \) and non-negative measurable functions \( \delta_1(t) \) and \( \delta_2(t) \) such that the function \( M(t,2\delta_1(t),2\delta_2(t)) \) is integrable over the domain \( T \) and such that for almost all \( t \) in \( T \) we have

\[ M(t,2x,2y) \leq KM(t,x,y) \]  \hspace{1cm} (1)

for all \( x \) and \( y \) satisfying \( |x| \geq \delta_1(t) \) and \( |y| \geq \delta_2(t) \).

We say that a GN*-function satisfies a \( \Delta_0 - \) condition if it satisfies a \( \Delta - \) condition with \( \delta_1(t) = 0 \) and \( \delta_2(t) = 0 \) for almost all \( t \) in \( T \).

In Definition 1.5 we could have used any constant \( \tau > 1 \) in place of the scalar 2 in (1).

**Definition 1.6:**

We say that a GN'-function \( \overline{M}(t,x_1, x_2, \ldots, x_n) \) satisfies a \( \Delta - \) condition if there exists a constant \( K \geq 2 \) and non-negative measurable functions \( \delta_1(t), \delta_2(t), \ldots, \)
$\delta_n(t)$ such that the function $M(t, 2\delta_1(t), 2\delta_2(t), \ldots, 2\delta_n(t))$ is integrable over the domain $T$ and such that for almost all $t$ in $T$ we have

$$M(t, 2x_1, 2x_2, \ldots, 2x_n) \leq KM(t, x_1, x_2, \ldots, x_n)$$

(1)

for all $x_1, x_2, \ldots, x_n$ satisfying $|x_1| \geq \delta_1(t), |x_2| \geq \delta_2(t), \ldots, |x_n| \geq \delta_n(t)$.

Thus, according to this definition, the statement above can be formulated as:

We say that a GN'-function satisfies a $\Delta_0 -$condition if it satisfies a $\Delta -$condition with $\delta_1(t) = 0, \delta_2(t) = 0, \ldots, \delta_n(t) = 0$ for almost all $t$ in $T$.

In Definition 1.6 we could have used any constant $\tau > 1$ in place of the scalar 2 in (1).

**Definition 1.7** Hassen (2007)

For each $t$ in $T$ and $h > 0$ let

$$M_h(t, x, y) = \int_{E^n} \int_{E^n} M(t, x + z, y + w)J_h(z)J_h(w)dzdw,$$

where $J_h(z)$ and $J_h(w)$ are non-negative, $C^\infty$ function with compact support in a ball of a radius $h$ such that $\int_{E^n} \int_{E^n} J_h(z)J_h(w)dtdw = 1$.

Moreover, let $x_0$ and $y_0$ are any tow points (depending on $h$, $t$) which satisfy the inequality

$$M_h(t, x_0, y_0) \leq M_h(t, x, y)$$

for all $x$ and $y$ in $E^n$. Then the function $\hat{M}_h(t, x, y)$ defined for each $t$ in $T$ and $h > 0$ by

$$\hat{M}_h(t, x, y) = M_h(t, x + x_0, y + y_0) - M_h(t, x_0, y_0)$$

is called a **mean function** for $M(t, x, y)$ relative to the minimizing point $x_0$ and $y_0$.

**Definition 1.8:**

For each $t$ in $T$ and $h > 0$ let
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\[ M_h(t, x_1, x_2, \ldots, x_n) = \int \int \ldots \int M(t, x + y_1, x + y_2, \ldots, x + y_n) J_h(y_1) J_h(y_2) \ldots J_h(y_n) dy_1 dy_2 \ldots dy_n \]

where \( J_h(y_1), J_h(y_2), \ldots, J_h(y_n) \) are no negative, \( c^\infty \) function with compact support in a ball of a radius \( h \) such that

\[ \int \int \ldots \int J_h(y_1) J_h(y_2) \ldots J_h(y_n) dt dt \ldots dt = 1. \]

Moreover, let \( x_{01}, x_{02}, \ldots, x_{0n} \) are any points (depending on \( h, t \)) which satisfy the inequality

\[ M_h(t, x_{01}, x_{02}, \ldots, x_{0n}) \leq M_h(t, x_1, x_2, \ldots, x_n) \]

for all \( x_1, x_2, \ldots, x_n \) in \( E^n \). Then the function \( \hat{M}_h(t, x_1, x_2, \ldots, x_n) \) defined for each \( t \) in \( T \) and \( h > 0 \) by

\[ \hat{M}_h(t, x_1, x_2, \ldots, x_n) = M_h(t, x_1 + x_{01}, x_2 + x_{02}, \ldots, x_n + x_{0n}) - M_h(t, x_{01}, x_{02}, \ldots, x_{0n}) \]

is called a mean function for \( M(t, x_1, x_2, \ldots, x_n) \) relative to the minimizing points \( x_{01}, x_{02}, \ldots, x_{0n} \).

**Theorem 1.9** Hassen (2007)

If \( M(t, x, y) \) is a GN*-function for which \( \bar{M}(t, c, c') \) is integrable in \( t \) for each \( c \) and \( c' \), then \( \hat{M}_h(t, x, y) \) is a GN*-function.

**Theorem 1.10** Hassen (2007)

If \( M(t, x, y) \) is a GN*-function satisfying a \( \Delta \)-condition and for which \( \bar{M}(t, c, c') \) is integrable in \( t \) for each \( c \) and \( c' \), then \( \hat{M}_h(t, x, y) \) satisfies a \( \Delta \)-condition.

**Theorem 1.11** Hassen (2007)

For each \( h > 0 \) let \( x_0^h \) and \( y_0^h \) be the minimizing point of \( M_h(t, x, y) \)
defining $\hat{M}_h(t,x,y)$. Then for each $t$ in $T$ and each $x, y$ in $E^n$, there exists $K(t,x,y)$ such that

$$
\lim_{h \to 0} \hat{M}_h(t,x,y) = M(t,x,y) + K(t,x,y) \lim_{h \to 0} |x^h_0| \lim_{h \to 0} |y^h_0|
$$

**Corollary 1.12 Hassen (2007)**

Suppose $M(t,x,y)$ is a GN*-function such that $M(t,x,y) = M(t,-x,-y)$. Then for each $t$ in $T$ and $x, y$ in $E^n$, we have

$$
\lim_{h \to 0} M_h(t,x,y) = \hat{M}(t,x,y)
$$

**Theorem 1.13 Hassen (2007)**

The sets $B$ and $A_h$ are closed convex sets.

**Theorem 1.14 Hassen (2007)**

Let $B_e = \{(x, y) : M(t, x, y) < e\}$ for each $t$ in $T$. Then for given any $e > 0$, there is a constant $h_0 > 0$, such that $A_h \subset B_e$ for each $h \leq h_0$.

**Theorem 1.15 Hassen (2007)**

Suppose $M(t,x,y)$ is a GN*-function which is strictly convex in $x$ and $y$ for each $t$. Then $h, A_h = \{(0,0)\}$ for each $h$.

**Theorem 1.16 Hassen (2013)**

A necessary and sufficient condition that (1.5.1) holds is that if

$$
\left| x_1 \right| \leq \left| y_1 \right|, \left| x_2 \right| \leq \left| y_2 \right|, ..., \left| x_n \right| \leq \left| y_n \right|
$$

then there exists constants $K \geq 1, d_1 \geq 0, d_2 \geq 0$.
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\[ \ldots, d_n \geq 0 \text{ such that } M(t, x_1, x_2, \ldots, x_n) \leq KM(t, y_1, y_2, \ldots, y_n) \text{ for each } t \text{ in } T, \]

\[ |x_1| \geq d_1, |x_2| \geq d_2, \ldots, |x_n| \geq d_n \]

**Theorem 1.17 Hassen (2010)**

A GN'-function \( M(t, x_1, x_2, \ldots, x_n) \) satisfies a \( \Delta \)-condition if and only if given any \( \tau > 1 \) there exists a constant \( K_\tau \geq 2 \) and a non-negative measurable functions \( \delta_1(t), \delta_2(t), \ldots, \delta_n(t) \) such that \( M(t, 2\delta_1(t), 2\delta_2(t), \ldots, 2\delta_n(t)) \) is integrable over \( T \) and such that for almost all \( t \) in \( T \) we have

\[ M(t, \alpha_1, \alpha_2, \ldots, \alpha_n) \leq K_\tau M(t, x_1, x_2, \ldots, x_n), \quad (1) \]

whenever \( |x_1| \geq \delta_1(t) \), \( |x_2| \geq \delta_2(t) \), \ldots, \( |x_n| \geq \delta_n(t) \).

2. Generalized mean function:

**Theorem 2.1:**

If \( M(t, x_1, x_2, \ldots, x_n) \) is a GN'-function for which \( M(t, c_1, c_2, \ldots, c_n) \) is integrable in \( t \) for each \( c_1, c_2, \ldots, c_n \), then \( \hat{M}_h(t, x_1, x_2, \ldots, x_n) \) is a GN'-function.

**Proof:**

We will show this result by justifying conditions (i)-(iv) of the definition 3.1.1. By hypothesis and the choice of \( x_{01}, x_{02}, \ldots, x_{0n} \), we have for each \( h, \)

\[ \hat{M}_h(t, x_1, x_2, \ldots, x_n) \geq 0 \quad \text{and} \quad \hat{M}_h(t,0,0,\ldots,0) = 0. \]

On the other hand, if \( x_1 \neq 0, x_2 \neq 0, \ldots, x_n \neq 0 \), then \( M(t, x_1, x_2, \ldots, x_n) > 0 \), and hence there are constants \( h_{01}, h_{02}, \ldots, h_{0n} \) such that

\[ a = \inf_{w} M(t, x_1 + w_1, x_2 + w_2, \ldots, x_n + w_n) > 0 \]
However, since $M(t, x_1, x_2, \ldots, x_n) = 0$ if and only if $x_1 = 0, x_2 = 0, \ldots, x_n = 0$, the minimizing points $x_{o1}$ tends to zero, $x_{o2}$ tends to zero, ..., $x_{on}$ tends to zero as $h$ tends to zero. Therefore, we can choose $g_{o1} \leq h_{o1}, g_{o2} \leq h_{o2}, \ldots, g_{on} \leq h_{on}$ such that if $h \leq g_{o1}, h \leq g_{o2}, \ldots, h \leq g_{on}$ then $M(t, x_{o1} + y_{o1}, x_{o2} + y_{o2}, \ldots, x_{on} + y_{on}) < a$ for all $y_{o1}, y_{o2}, \ldots, y_{on}$ for which $|x_{o1} + y_{o1}| < h, |x_{o2} + y_{o2}| < h, \ldots, |x_{on} + y_{on}| < h$ for this $g_{o1}, g_{o2}, \ldots, g_{on}$ we obtain the inequality

$$M(t, x_1 + x_{o1} + x_{o2} + y_{o2}, \ldots, x_n + x_{on} + y_{on}) \geq \inf_{w \leq g_{o1}} \left[ M(t, x_1 + w, x_2 + w, \ldots, x_n + w) \right] \geq a$$

$$> M(t, x_{o1} + y_{o1}, x_{o2} + y_{o2}, \ldots, x_{on} + y_{on})$$

whenever $|x_{o1} + y_{o1}| \leq g_{o1}, |x_{o2} + y_{o2}| \leq g_{o2}, \ldots, |x_{on} + y_{on}| \leq g_{on}. This means for some $h \leq g_{o1}, h \leq g_{o2}, \ldots, h \leq g_{on}$ we have

$$M(t, x_1 + x_{o1} + y_{o1}, x_2 + x_{o2} + y_{o2}, \ldots, x_n + x_{on} + y_{on}) > M(t, x_{o1} + y_{o1}, x_{o2} + y_{o2}, \ldots, x_{on} + y_{on})$$

$$M_h(t, x_1 + x_{o1}, x_2 + x_{o2}, \ldots, x_n + x_{on}) > M_h(t, x_{o1}, x_{o2}, \ldots, x_{on})$$

or $\hat{M}_h(t, x_1, x_2, \ldots, x_n) > 0$ if $x_1 \neq 0, x_2 \neq 0, \ldots, x_n \neq 0$ which proves property (i).

Properties (ii) and (iii) for $\hat{M}_h(t, x_1, x_2, \ldots, x_n)$ follow easily from the same properties for $M(t, x_1, x_2, \ldots, x_n)$. Let us now show (iv). By assumption, there are constants $d_1 \geq 0, d_2 \geq 0, \ldots, d_n \geq 0$ such that

$$\tau(t)\overline{M}(t, c_1, c_2, \ldots, c_n) \leq \overline{M}(t, c_1, c_2, \ldots, c_n) \leq \overline{M}(t, c_1, c_2, \ldots, c_n)$$

(1)
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for all \( c_1 \geq d_1, c_2 \geq d_2, \ldots, c_n \geq d_n \). Furthermore, it is not difficult to show that for all \( c \) and \( c' \) we have

\[
\bar{M}(t, c_1, c_2, \ldots, c_n) \geq \sup_{1 \leq i \leq n} M(t, x_i, x_i, \ldots, x_i) \quad (2)
\]

and for some fixed \( w_1, w_2, \ldots, w_n \)

\[
\inf_{1 \leq i \leq n} M(t, x_i + w_i \cdot x_i, x_i + w_i \cdot x_i, \ldots, x_i + w_i \cdot x_i) \leq \inf_{1 \leq i \leq n} M(t, x_i + w_i, x_i + w_i, \ldots, x_i + w_i) \quad (3)
\]

By using (2), we obtain (for each \( t \) in \( T \)) that

\[
\tau(t) \sup_{1 \leq i \leq n} M(t, w_i, w_i, \ldots, w_i) \leq \tau(t) \sup_{1 \leq i \leq n} M(t, r_i, r_i, \ldots, r_i) \leq \sup_{1 \leq i \leq n} M(t, r_i, r_i, \ldots, r_i) \quad (4)
\]

where \( w_i = x_i + x_{0i} + r_i \) for \( i = 1 \) to \( n \). On the other hand, by (1) and (3), we achieve

\[
\tau(t) \sup_{1 \leq i \leq n} M(t, w_i, w_i, \ldots, w_i) \leq \inf_{1 \leq i \leq n} M(t, w_i, w_i, \ldots, w_i) \leq \inf_{1 \leq i \leq n} M(t, x_i + x_{0i} + r_i, x_i + x_{0i} + r_i, \ldots, x_i + x_{0i} + r_i) < \inf_{1 \leq i \leq n} M(t, x_i + x_{0i} + r_i, x_i + x_{0i} + r_i, \ldots, x_i + x_{0i} + r_i) \quad (5)
\]
If we combine (4) and (5), then for all \( c_i \geq d_i \) for \( i = 1 \) to \( n \) and we arrive at

\[
\tau(t) \sup \frac{M(t, x_1 + x_{01} + r_1, x_2 + x_{02} + r_2, \ldots, x_n + x_{0n} + r_n)}{n} \leq \inf \frac{M(t, x_1 + x_{01} + r_1, x_2 + x_{02} + r_2, \ldots, x_n + x_{0n} + r_n)}{n}
\]

for \( 1 \leq i \leq n \)

From this inequality, we obtain

\[
\inf \frac{\hat{M}_h(t, x, x, \ldots, x)}{n} \geq \int \ldots \int \inf \frac{M(t, x + x + r, x + x + r, \ldots, x + x + r)}{n}
\]

for \( 1 \leq i \leq n \)

\[
1 \leq i \leq n
\]

\[
- M(t, x_1 + x_{01} + r_1, x_2 + x_{02} + r_2, \ldots, x_n + x_{0n} + r_n) J_h(r) J_h(r) \ldots J_h(r) dr \ldots dr
\]

\[
\geq \int \ldots \int \{\tau(t) \sup \frac{M(t, x + x + r, x + x + r, \ldots, x + x + r)}{n}
\]

for \( 1 \leq i \leq n \)

\[
M(t, x_1 + x_{01} + r_1, x_2 + x_{02} + r_2, \ldots, x_n + x_{0n} + r_n) J_h(r) J_h(r) \ldots J_h(r) dr \ldots dr
\]

and

\[
\sup \frac{\hat{M}_h(t, x, x, \ldots, x)}{n} \leq \int \ldots \int \sup M(t, x_1 + x_{01} + r_1, x_2 + x_{02} + r_2, \ldots, x_n + x_{0n} + r_n) J_h(r) J_h(r) \ldots J_h(r) dr \ldots dr
\]

Moreover, since \( \lim_{c_i \to \infty} \sup \frac{M(t, x_1 + x_{01} + r_1, x_2 + x_{02} + r_2, \ldots, x_n + x_{0n} + r_n)}{n} = \infty \)

for fixed \( x_{0i}, r_i \) for \( 1 \leq i \leq n \) such that \( |r_i| \leq h_i \) for \( 1 \leq i \leq n \) given
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\[ K_1(t) = 2\sup_{\|r\| \leq h} M(t, x_{01} + r_1, x_{02} + r_2, \ldots, x_{0n} + r_n) / \inf_{\|\tau\|} \tau(t) \]

there are \(d_i > 0, 1 \leq i \leq n\) such that if \(c_i \geq d_i, 1 \leq i \leq n\), then

\[ \sup_{\|x\| = c_i} \frac{M(t, x_1 + x_{01} + r_1, x_2 + x_{02} + r_2, \ldots, x_n + x_{0n} + r_n)}{h} \geq K_1. \]

Therefore, by using (3.3.8) and (3.3.9), we achieve the inequalities

\[ \inf_{\|x\| = c_i} \frac{\hat{\Delta}_i(t, x, x_1, x_2, \ldots, x_n)}{h} \geq \tau(t) - \frac{1}{2} \inf_{\|\tau\|} \tau(t) \]

\[(8)\]

for all \(c_i \geq d_{0i} = \max(d_i, d_i', x_{0i})\). Taking the infimum of both sides of (2.1.8) over \(t\), shows the first part of the property (iv). To show the latter part, assume \(d_{0i} > 0, 1 \leq i \leq n\) and then \(\sup_{\|x\| = d_{0i}} \hat{\Delta}_i(t, x_1, x_2, \ldots, x_n)\) is integrable over \(t\) in \(T\) since

\[ \inf_{\|x\| = d_{0i}} \sup_{\|\tau\|} M(t, x_1 + x_{01} + r_1, x_2 + x_{02} + r_2, \ldots, x_n + x_{0n} + r_n) \geq \tau(t) - \frac{1}{2} \inf_{\|\tau\|} \tau(t) \]

it is bounded by the integrable function \(\hat{\Delta}_i(t, x_1, x_2, \ldots, x_n)\) where \(d_{i2} = d_{0i} + |x_{0i}| + h\). This proves property (iv) and the theorem. ■

In the next theorem we show under what condition \(\hat{\Delta}_i(t, x_1, x_2, \ldots, x_n)\) satisfies a \(\Delta - \) condition.
Theorem 2.2: 

If $M(t, x_1, x_2, \ldots, x_n)$ is a GN'-function satisfying a $\Delta-$condition and for which $\overline{M}(t, c_1, c_2, \ldots, c_n)$ is integrable in $t$ for each $c_1, c_2, \ldots, c_n$ then $\hat{M}_h(t, x_1, x_2, \ldots, x_n)$ satisfies a $\Delta-$condition.

Proof:

It suffices to show that $M_h(t, x_1, x_2, \ldots, x_n)$ satisfies a $\Delta-$condition.

For, $\hat{M}_h(t, x_1, x_2, \ldots, x_n)$ is the sum of a constant and a translation of $M_h(t, x_1, x_2, \ldots, x_n)$ and neither of these operations affects the growth condition. Let us observe first that if $|x_i| \geq 2$ for $1 \leq i \leq n, |z_i| \leq h_1$ for $1 \leq i \leq n$ then $|2x_i + z_i| \leq |3x_i + z_i|$ for $1 \leq i \leq n$. Hence, by Theorem (1.16), there are constants $K \geq 1$ and $d_1 \geq 0$ such that

$$M_h(t, 2x_1, 2x_2, \ldots, 2x_n) \leq k \int \ldots \int_{E^n} M(t, 3(x_1 + z_1), 3(x_2 + z_2), \ldots, 3(x_n + z_n))$$

$$K_3 M_h(t, x_1, x_2, \ldots, x_n) \int_{h_1} J_h(z_1) \int_{h_2} J_h(z_2) \ldots \int_{h_n} J_h(z_n) dz_1 dz_2 \ldots dz_n$$

for all $x_i$ for $1 \leq i \leq n$ such that $|x_i| \geq d_2$ for $1 \leq i \leq n$ and $d_2 = \max(d_1, 2)$. On the other hand, by theorem (1.17),

$$\int \ldots \int_{E^n} M(t, 3(x_1 + z_1), 3(x_2 + z_2), \ldots, 3(x_n + z_n)) J_h(z_1) J_h(z_2) \ldots J_h(z_n) dz_1 dz_2 \ldots dz_n \leq$$

$$\int \ldots \int_{E^n} M_h(t, x_1, x_2, \ldots, x_n)$$

there is a constant $K_3 \geq 2, \delta_i(t) \geq 0$ for $1 \leq i \leq n$ such that for almost all $t$ in $T$ for all $x_i, z_i$ for $1 \leq i \leq n$ such that $|x_i + z_i| \geq \delta_i(t)$ for $1 \leq i \leq n$ where $|z_i| \leq h_i$ for $1 \leq i \leq n$.

By combining the above two inequalities, we achieve

$$M_h(t, 2x_1, 2x_2, \ldots, 2x_n) \leq KK_3 M_h(t, x_1, x_2, \ldots, x_n)$$
for all $|x_i| > \max(d_{2i}, \delta_i(t) + h) = \delta_i'(t)$ Since $\overline{M}(t, 2\delta_1(t), 2\delta_2(t), \ldots, 2\delta_n(t))$ is integrable over $T$, this yields the integrability of $\overline{M}_h(t, 2\delta_1'(t), 2\delta_2'(t), \ldots, 2\delta_n'(t))$ which proves the theorem. ■

For each $t$ in $T$ and $x_1, x_2, \ldots, x_n$ in $E^n$ it is known that

$$
\lim_{h \to 0} M_h(t, x_1, x_2, \ldots, x_n) = M(t, x_1, x_2, \ldots, x_n).
$$

However, the same property does not hold in general for $\hat{M}_h(t, x_1, x_2, \ldots, x_n)$. This is the point of the next theorem.

**Theorem 2.3**:

For each $h > 0$ let $x_i^h$ for $1 \leq i \leq n$ be the minimizing point of $M_h(t, x_1, x_2, \ldots, x_n)$ defining $\hat{M}_h(t, x_1, x_2, \ldots, x_n)$. Then for each $t$ in $T$ and each $x_i$ for $1 \leq i \leq n$ in $E^n$, there exists $K(t, x_1, x_2, \ldots, x_n)$ such that

$$
\lim_{h \to 0} \hat{M}_h(t, x_1, x_2, \ldots, x_n) = M(t, x_1, x_2, \ldots, x_n) + K(t, x_1, x_2, \ldots, x_n) \prod_{i=1}^{n} \lim_{h \to 0} x_i^h
$$

**Proof:**

By the definition of $\hat{M}_h(t, x_1, x_2, \ldots, x_n)$ we can write

$$
\left| \hat{M}_h(t, x_1, x_2, \ldots, x_n) - M(t, x_1, x_2, \ldots, x_n) \right| \leq \int_{E^n} \cdots \int_{E^n} M(t, x_1 + x_0^h + z_1, x_2 + x_0^h + z_2, \ldots, x_n + x_0^h + z_n) - M(t, x_1, x_2, \ldots, x_n) \left| J_h(z_1)J_h(z_2)\ldots J_h(z_n)dz_1dz_2\ldots dz_n \right|
$$

(1)
However, we know that
\[
M(t, x_1 + x_0^h + z_1, x_2 + x_0^h + z_2, ..., x_n + x_0^h + z_n) - M(t, x_1^h, x_2^h, ..., x_n^h) \leq M(t, x_1 + x_0^h + z_1, x_2 + x_0^h + z_2, ..., x_n + x_0^h + z_n - M(t, x_1^h, x_2^h, ..., x_n^h) \leq M(t, x_1 + x_0^h + z_1, x_2 + x_0^h + z_2, ..., x_n + x_0^h + z_n) - M(t, z_1^h, z_2^h, ..., z_n^h).
\]

Moreover, since \( M(t, x_1, x_2, ..., x_n) \) is a convex function, it satisfies a Lipschitz condition on compact subsets of \( E^n \) (see[Skaff (1968), Th.5.1]). Therefore, there exists \( K_1(t, x_1, x_2, ..., x_n) \) and \( K_2(t, x_1, x_2, ..., x_n) \) such that
\[
M(t, x_1 + x_0^h + z_1, x_2 + x_0^h + z_2, ..., x_n + x_0^h + z_n) - M(t, x_1^h, x_2^h, ..., x_n^h) \leq K_1(t, x_1, x_2, ..., x_n) \| x_0^h \| \| z_1 \| \| x_0^h \| \| z_2 \| \| x_0^h \| \| z_n \|.
\]

and
\[
M(t, x_0^h + z_1, x_0^h + z_2, ..., x_0^h + z_n) - M(t, z_1^h, z_2^h, ..., z_n^h) \leq K_2(t, x_1, x_2, ..., x_n) \| x_0^h \| \| x_0^h \| \| x_0^h \|.
\]

If we combine (3) and (4) with (2) and if we substitute the resulting expression into (1), we achieve the inequality
\[
\begin{align*}
\left| \bar{M}_h(t, x_1, x_2, ..., x_n) - M(t, x_1, x_2, ..., x_n) \right| & \leq \sum_{i=1}^{n} \left| x_0^h \left| K_i(t, x_1, x_2, ..., x_n) \right| \right| J_h(z_i) J_h(z_i) ... J_h(z_i) dz_1 dz_2 ... dz_n \\
& + \sum_{i=1}^{n} \left| K_i(t, x_1, x_2, ..., x_n) \right| J_h(z_i) J_h(z_i) ... J_h(z_i) dz_1 dz_2 ... dz_n \\
& + \sum_{i=1}^{n} \left| M(t, z_1, z_2, ..., z_n) J_h(z_i) J_h(z_i) ... J_h(z_i) dz_1 dz_2 ... dz_n \\
& + \sum_{i=1}^{n} \left| M(t, z_1, z_2, ..., z_n) J_h(z_i) J_h(z_i) ... J_h(z_i) dz_1 dz_2 ... dz_n \\
& + \sum_{i=1}^{n} \left| M(t, z_1, z_2, ..., z_n) J_h(z_i) J_h(z_i) ... J_h(z_i) dz_1 dz_2 ... dz_n \right|.
\end{align*}
\]
Generalized mean function for n-variable

Since the last four integrals on the right side tend to zero as $h$ tends to zero, we prove the theorem by setting

$$K(t,x_1,x_2,\ldots,x_n) = K_1(t,x_1,x_2,\ldots,x_n) + K_2(t,x_1,x_2,\ldots,x_n)$$

**Corollary 2.4:**

Suppose $M(t,x_1,x_2,\ldots,x_n)$ is a GN'-function such that

$$M(t,x_1,x_2,\ldots,x_n) = M(t,-x_1,-x_2,\ldots,-x_n).$$

Then for each $t$ in $T$ and $x_i$ in $E^n$ for $i=1$ to $n$, we have

$$\lim_{h \to 0} M_h(t,x_1,x_2,\ldots,x_n) = \hat{M}(t,x_1,x_2,\ldots,x_n)$$

**Proof:**

This result is clear since $\lim_{h \to 0} |x_i^h| = 0$ for $i=1$ to $n$ if $M((t,x_1,x_2,\ldots,x_n)) = M(t,-x_1,-x_2,\ldots,-x_n)$. In fact, if $M(t,x_1,x_2,\ldots,x_n)$ is even in $(x_1,x_2,\ldots,x_n)$ then the $x_i^h = 0$ for $i=1$ to $n$ for all $h$.

For each $t$ in $T$ let $A_h$ denote the set of minimizing points of

$$M_h(t,x_1,x_2,\ldots,x_n)$$

and let $B$ represents the null space of $M(t,x_1,x_2,\ldots,x_n)$ relative to points in $E^n \times E^n \times \ldots \times E^n$, i.e.,

$$B = \{(x_1,x_2,\ldots,x_n) \in E^n \times E^n \times \ldots \times E^n : M(t,x_1,x_2,\ldots,x_n) = 0\}.$$ 

If $M(t,x_1,x_2,\ldots,x_n)$ is a GN'-function, then $B = \{(0,0,\ldots,0)\}$. For the sake of argument, let us suppose that $M(t,x_1,x_2,\ldots,x_n)$ has all the properties of a GN'-function except that $M(t,x_1,x_2,\ldots,x_n) = 0$ need not imply $x_i = 0$ for $i=1$ to $n$. 

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We will show the relationships that exist between $A_h$ and $B$. This is the content of the next few theorems.

**Theorem 2.5:**

The sets $B$ and $A_h$ are closed convex sets.

**Proof:**

This result follows from the convexity and continuity of $M(t,x_1,x_2,...,x_n)$ in $x_i$ for $i=1$ to $n$ for each $t$ in $T$. ■

**Theorem 2.6:**

Let $B_e = \{(x_1,x_2,...,x_n) : M(t,x_1,x_2,...,x_n) < e\}$ for each $t$ in $T$. Then given any $e > 0$, there is a constant $h_0 > 0$ such that $A_h \subseteq B_e$ for each $h \leq h_0$.

**Proof:**

Since $B \subseteq B_e$, we can choose $h_0$ sufficiently small so that if $(x_1,x_2,...,x_n)$ is in $B$ then $(x_1 + z_1,x_2 + z_2,...,x_n + z_n)$ is in $B_e$ for all $(z_1,z_2,...,z_n)$ such that $|z_i| \leq h_0$ for $i=1$ to $n$. Let $(z_{01},z_{02},...,z_{0n})$ be arbitrary but fixed points in $A_h$, $h \leq h_0$. Then

$$M_h(t,z_{01},z_{02},...,z_{0n}) \leq M_h(t,x_1,x_2,...,x_n) \text{ for all } x_i \text{ for } i=1 \text{ to } n.$$

Therefore, if $(x_1,x_2,...,x_n)$ in $B$, we have $M_h(t,z_{01},z_{02},...,z_{0n}) < e$ by our choice of $h_0$. Letting $h$ tend to zero yields $M(t,z_{01},z_{02},...,z_{0n}) < e$, i.e., $(z_{01},z_{02},...,z_{0n})$ in $B_e$.

We have commented above that $A_h = \{(0,0,...,0)\}$
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\[ M(t,x_1,x_2,...,x_n) = M(t,-x_1,-x_2,...,-x_n). \]

It is also true if \( M(t,x_1,x_2,...,x_n) \) is strictly convex in \( x \) for each \( t \) in \( T \).

**Theorem 2.7:**

Suppose \( M(t,x_1,x_2,...,x_n) \) is a GN*-function which is strictly convex in \( x_i \) for each \( t \). Then \( h, A_h = \{(0,0,...,0)\} \) for each \( h \).

**Proof:**

Suppose that there exists \( z_{0i} \neq x_{0i} \) for \( 1 \leq i \leq n \) such that \( z_{0i}, x_{0i} \) for \( 1 \leq i \leq n \) are in \( A_h \). Let \( z_i = \frac{(x_{0i} + z_{0i})}{2} \) for \( 1 \leq i \leq n \). Then, since \( M(t,x_1,x_2,...,x_n) \) is strictly convex, \( M_h(t,x_1,x_2,...,x_n) \) is strictly convex in \( x_1,x_2,...,x_n \), therefore, we have

\[
M_h(t,z_1,z_2,...,z_n) = \frac{1}{2} M_h(t,x_1,x_2,...,x_n) + \frac{1}{2} M_h(t,z_1,z_2,...,z_n). \tag{1}
\]

However, \( (x_{01}, x_{02},...,x_{0n}), (z_{01}, z_{02},...,z_{0n}) \) are in \( A_h \) reduces (1) to the inequality \( M_h(t,z_1,z_2,...,z_n) < M_h(t,x_1,x_2,...,x_n) \) for all \( x_i \) for \( i = 1 \) to \( n \).

This means \( z_1, z_2, ... \) and \( z_n \) are in \( A_h \) and are \( (x_{01}, x_{02},...,x_{0n}), (z_{01}, z_{02},...,z_{0n}) \) not in \( A_h \) which is a contradiction. Hence, \( x_{0i} = z_{0i} \) for \( i = 1 \) to \( n \). Since \( M(t,x_1,x_2,...,x_n) \) is a GN'-function, \( B = \{(0,0,...,0)\} \). In this case \( x_{0i} = z_{0i} = 0 \) for \( i = 1 \) to \( n \).

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