On Geometry of Viasman-Gray Manifold

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Abstract. In this paper, we studied the geometric structure of one important class of almost Hermitian manifold which is called Viasman-Gray manifold. This manifold is a generalization of the classes nearly Kahler manifold and the locally conformal Kahler manifold. We proved that, if $M$ is Viasman-Gray manifold with flat conformal curvature tensor, then $M$ is a manifold of class $R_1$ if and only if $M$ is a manifold of flat Ricci tensor. The necessary condition that $M$ is of zero scalar curvature tensor has been found. Finally, we proved that, if $M$ is $VG$-manifold of class $W_1$ and of flat Ricci tensor then $M$ is Kahler manifold.

Keywords: Almost Hermitian manifold, Viasman-Gray manifold, conformal curvature tensor.

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1 Introduction.

The study of conformal invariant properties of Riemannian manifold with other structure play an important place in modern geometry. This study began by the work of Gray and Hervalla [2] which were classified the almost Hermitian manifold into sixteen classes, between these classes there are eight invariant under the conformal transformation metric. One of these classes is Viasman-Gray manifold that is denoted by $W_1 \oplus W_4$ which is represents a generalization of the classes $W_1$ and $W_4$. The class $W_1$ is called a nearly Kahler manifold. This class defined by Tachibana [9] under the name $K$-space which is not invariant under conformal transformation metric but it belongs to the class locally conformal nearly Kahler manifold which is invariant under conformal transformation. The class $W_4$ is called a locally conformal Kahler manifold which is invariant with respect to the transformation metric.
The class $W_3 \oplus W_4$ has rich differential geometric properties and it is represents interesting study. So there are many researchers are studied this class, in particular Gray and Vanheke\([3]\), Viasman\([11]\) and Kirichenko \([6]\), \([7]\).

2Preliminaries.

Let $M$ be a smooth manifold of dimension $2n(n > 1), C^\infty(X)$ be an algebra of smooth functions on $M$, $X(M)$ be a Lie algebra of vector fields on $M$. An almost Hermitian structure ($AH-$structure) on $M$ is a pair of tensors $\{J, g =<>\}$, where $J$ is an almost complex structure, $g =<>$ is a Riemannian metric, such that $<JX, JY> = <X, Y>$; $X, Y \in X(M)$. A smooth manifold $M$ with $AH-$structure is called an almost Hermitian manifold ($AH-$manifold).

In the tangent space $T_p(M)$ there exist a basis of the form $\{\varepsilon_1, \ldots, \varepsilon_n, \bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_n\}$. Its corresponding frame is $\{p, \varepsilon_1, \ldots, \varepsilon_n, \bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_n\}$. Suppose that the indexes $i, j, k, l$ in the range 1, 2, ..., $2n$ and the indices $a, b, c, d, e, f, g, h$ in the range 1, 2, ..., $n$. Denote $\hat{a} = a + n$.

It is known \([5]\) that the setting an $AH-$structure on $M$ is equivalent to the setting of an $G-$structure in the principle fiber bundle of all complex frames of manifold $M$ which contains $G$-structure group that is the unitary group $U(n)$ which is called an adjoined $G$-structure. In the space of the adjoined $G$-structure, the following forms define matrices which give components of tensor fields $g$ and $J$:

\[
(g_{ij}) = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}
\]

\[
(J^i_j) = \begin{bmatrix} \sqrt{-1}I_n & 0 \\ 0 & -\sqrt{-1}I_n \end{bmatrix},
\]

where $I_n$ is the unit matrix of order $n$.

Recall that \([4]\) an $AH-$structure $(J, g =<>)$ is called a structure of class $W_3 \oplus W_4$ or Viasman–Gray structure if,

\[
\nabla(F)(X, Y) = \frac{-1}{2(n-1)}\{X, Y \delta F(Y) - <X, Y \delta F(X), -JX, JY \delta F(JX)>\},
\]

where $\nabla$ is the Riemannian connection of $g$, $F(X, Y) =<JX, JY>$ is the Kahler form, $\delta$ is a codervative and $X, Y \in X(M)$.

An $AH-$structure $(J, g = <>)$ is called a structure of class $W_4$ or nearly Kahler if its Kahler form is a Killing form, or equivalently,

\[
\nabla_X(J) = 0, X \in X(M).
\]

An $AH-$structure $(J, g =<, >)$ is called a structure of class $W_4$ if
A manifold \( M \) with Viasman-Gray structure is called a Viasman-Gray manifold (\( VG \)-manifold). For each \( AH \)-manifold, in particular for \( VG \)-manifold defined a Lie form by the formula

\[
\alpha = \frac{1}{n-1} \delta F \circ J
\]

It is well known [4] that the structure equations of Riemannian connection of the \( VG \)-structure on adjoined \( G \)-structure space which are called the structure equations of \( VG \)-structure have the forms

1) \[ \omega^a = \omega^a_b \Lambda \omega^b + B^{abc} \omega^c \Lambda \omega^b + B^{abc} \omega_b \Lambda \omega_c; \]
2) \[ \omega_a = -\omega^b_a \Lambda \omega_b + B^{abc} \omega_c \Lambda \omega^b + B^{abc} \omega^b \Lambda \omega^c; \]
3) \[ \omega^a_b = \omega^a_c \Lambda \omega^b c + (B^{abc} \omega^b d + A^{bcd}) \omega^c \Lambda \omega^d + (B^{[bc]} \omega_d \Lambda \omega^d + A^{[bcd]} \omega^d \Lambda \omega^d), \]

where \( \{\omega^i\} \) ar the components of the solder form, \( \{\omega^i_j\} \) are the components of the connection form for Riemannian metric, \( \omega_a = \omega^a \) and \( \{A^{abcd}, A^{abc}, A^{bcd}\} \) are some functions on adjoined \( G \)-structure space. The functions \( \{A^{abcd}\} \) defined a tensor field on the manifold \( M \), this tensor field is called a tensor of holomorphic sectional curvature. It is known that \( A^{abcd} = A^{abc} \).

The tensors \( \{B^{abc}\} \) and \( \{B^{abc}\} \) are called the structure tensors and the tensors \( \{B^{abc}\} \) and \( \{B^{abc}\} \) are called the virtual tensors. It is obvious that \( B^{abc} = B^{abc} \) and \( B^{abc} = B^{abc} \).

**Remark 2.1.** By the Banaru’s classification of \( AH \)-manifold [1], the \( VG \)-manifold satisfies the following properties

\[
B^{[abc]} = B^{abc}; B_{[abc]} = B_{abc}; B^{[abc]} = \alpha^{[a}\delta^b_{c]}; B^{[abc]} = \alpha_{[a}\delta^b_{c]},
\]

where \( \{\alpha_a, \alpha^a \equiv \alpha^a_n\} \) are the components of the Lie form \( \alpha \).

**Lemma 2.2.** The components of Riemannian curvature tensor \( R \) of \( VG \)-Manifold in the adjoint \( G \)-structure space are given as the following forms:

1) \[ R_{abcd} = 2(B_{[abc]} + \alpha_{[a} B_{b]c}) ; \]
2) \[ R_{[abcd]} = 2A^{[abcd]} ; \]
3) \[ R_{[ab]cd} = 2(-B^{abc} B_{c[d} + \alpha^{[a}\delta^b_{d]}); \]
4) \[ R_{[abc]} = A^{[abcd]} + B^{[abcd]} B^{[abc]} - B^{[abc]} B^{[abc]}; \]

where \( \{\alpha_a, \alpha^a \equiv \alpha^a_n, \alpha_{ab}, \alpha^{[ab]} \} \) are some functions on adjoined \( G \)-structure space such that \( d\alpha_a + \alpha_a \omega^b_a = \alpha^b_a \omega_b + \alpha_{ab} \omega^b \) and
The others components of Riemannian curvature tensor $R$ can be obtained by the property of symmetry for $R$.

**Lemma 2.3[4].** The components of Ricci tensor of $V G$-manifold in adjoined $G$-structures space are given as the following forms:

1) $r_{ab} = \frac{1-n}{2} (\alpha_{ab} + \alpha_{ba} + \alpha_{a} \alpha_{b})$

2) $r_{\bar{a} \bar{b}} = 3 B c a h B c b h - A_{bc} a + \frac{n-1}{2} (\alpha^{a} \alpha_{b} - \alpha^{h} \alpha_{h}) - \frac{1}{2} \alpha^{h} \delta_{b}^{a} + (n-2) \alpha^{a} \alpha_{b}$

**Definition 2.4.** A scalar curvature tensor of an $A H$-manifold is denoted by $K$ and defined as:

$$K = g^{ij} r_{ij} \quad (2.5)$$

**Lemma 2.5.** For any almost Hermitian manifold. In the adjoined $G$-structure space, the scalar curvature tensor satisfies the following equation

$$K = 2r_{a}^{a}$$

**Proof.** In the adjoined $G$-structure space, the equation (2.5) becomes:

$$K = g^{ab} r_{ab} + g^{\bar{a} \bar{b}} r_{\bar{a} \bar{b}} + g^{ab} r_{ab} + g^{\bar{a} \bar{b}} r_{\bar{a} \bar{b}}$$

By using equation (2.1) we get:

$$K = 2g^{ab} r_{ab} = 2\delta_{a}^{b} r_{\bar{a} \bar{b}} = 2r_{a}^{a}$$

Therefore, $= 2r_{a}^{a}$.

3 Main results.

**Definition 3.1[8].** As for as the Riemannian space, the conformal curvature tensor or Weyl's tensor $\{W = W_{ijkl}^{i}\}$ of type (3,1) is defined by the form:

$$W_{ijkl} = R_{ijkl} + \frac{1}{m-2} \left( r_{ik} g_{jl} + r_{jl} g_{ik} - r_{il} g_{jk} - r_{jk} g_{il} \right) + \frac{K(g_{ij} g_{jk} - g_{ik} g_{jl})}{(m-2)(m-1)} \quad (3.1)$$

where $R_{ijkl}$ are the components of the Riemannian curvature tensor, $r_{ij}$ are the components of Ricci tensor, $g_{ij}$ are components of the Riemannian metric $g$ and $K$ is the scalar curvature tensor. This tensor is invariant under conformal transformation metric.

According to our case, the $A H$-manifold which we have, $\text{dim} M = 2n$, then the conformal curvature tensor is redefined by the following form:

$$W_{ijkl} = R_{ijkl} + \frac{1}{2(n-1)} \left( r_{ik} g_{jl} + r_{jl} g_{ik} - r_{il} g_{jk} - r_{jk} g_{il} \right) + \frac{K(g_{ij} g_{jk} - g_{ik} g_{jl})}{2(n-1)(2n-1)} \quad (3.2)$$

This tensor has similar properties to those of the Riemannian curvature tensor.

**Lemma 3.2.** In the adjoined $G$-structure space, the components of the conformal
curvature tensor of the VG-manifold are given by the following forms:

1) \( W_{abcd} = R_{abcd} \);

2) \( W_{\bar{a}bc\bar{d}} = R_{\bar{a}bc\bar{d}} + \frac{1}{2(n-1)} (r_{bd} \delta_{c}^{a} - r_{bc} \delta_{d}^{a}) \);

3) \( W_{\bar{a}\bar{b}c\bar{d}} = R_{\bar{a}\bar{b}c\bar{d}} + \frac{2}{(n-1)} r_{[c}^{a} \delta_{d]}^{b} - \frac{K \delta_{cd}^{ab}}{2(n-1)(2n-1)} \);

4) \( W_{\bar{a}bc\bar{d}} = R_{\bar{a}bc\bar{d}} + \frac{1}{2(n-1)} (r_{c}^{a} \delta_{d}^{b} + r_{b}^{d} \delta_{c}^{a}) - \frac{K \delta_{cd}^{ab}}{2(n-1)(2n-1)} \), where \( \delta_{cd}^{ab} = \delta_{c}^{a} \delta_{d}^{b} - \delta_{c}^{b} \delta_{d}^{a} \).

**Proof.** 1) For \( a, b, c, \) and \( l = d \), the equation (3.2) becomes:

\[
W_{abcd} = R_{abcd} + \frac{1}{2(n-1)} (r_{ac} g_{bd} + r_{bd} g_{ac} - r_{ad} g_{bc} - r_{bc} g_{ad}) + \frac{K (g_{bc} g_{ad} - g_{bd} g_{ac})}{2(n-1)(2n-1)}
\]

According to the equation (2.1), we get that

\[
W_{abcd} = R_{abcd}
\]

2) For \( i = \bar{a}, j = b, k = c \) and \( l = d \), we have

\[
W_{\bar{a}bc\bar{d}} = R_{\bar{a}bc\bar{d}} + \frac{1}{2(n-1)} (r_{c}^{a} g_{b\bar{d}} + r_{b}^{d} g_{ac} - r_{a}^{d} g_{bc} - r_{bc} g_{a\bar{d}}) + \frac{K (g_{bc} g_{a\bar{d}} - g_{b\bar{d}} g_{ac})}{2(n-1)(2n-1)}
\]

\[
W_{\bar{a}\bar{b}c\bar{d}} = R_{\bar{a}\bar{b}c\bar{d}} + \frac{1}{2(n-1)} (r_{ac} g_{b\bar{d}} + r_{b\bar{d}} g_{ac} - r_{ad} g_{bc} - r_{bc} g_{a\bar{d}}) + \frac{K (g_{bc} g_{a\bar{d}} - g_{b\bar{d}} g_{ac})}{2(n-1)(2n-1)}
\]

3) For \( i = \bar{a}, j = \bar{b}, k = c \) and \( l = d \), we have

\[
W_{\bar{a}\bar{b}c\bar{d}} = R_{\bar{a}\bar{b}c\bar{d}} + \frac{1}{2(n-1)} (r_{a\bar{c}} g_{b\bar{d}} + r_{b\bar{d}} g_{a\bar{c}} - r_{a\bar{d}} g_{bc} - r_{bc} g_{a\bar{d}}) + \frac{K (g_{bc} g_{a\bar{d}} - g_{b\bar{d}} g_{ac})}{2(n-1)(2n-1)}
\]
Lemma 3.3 [10]. In the adjoined $G$-structure space, an $AH$- manifold is manifold of class:

\[ R_1 \text{ if and only if } R_{abcd} = R_{\bar{a}b\bar{c}d} = 0, \]

\[ R_2 \text{ if and only if } R_{abcd} = R_{\bar{a}b\bar{c}d} = 0, \]

\[ R_3 (RK\text{-manifold}) \text{ if and only if } R_{abcd} = 0. \]

**Theorem 3.4**. If $M$ is $VG$-manifold with flat conformal curvature tensor, then $M$ is a manifold of class $R_1$ if and only if $M$ is a manifold of flat Ricci tensor.

**Proof.** Suppose that $M$ is $VG$-manifold with flat conformal curvature tensor.

Making use of Lemma 3.2 we get:

\[ R_{\bar{a}b\bar{c}d} + \frac{1}{2(n-1)}(r_{bd}\delta_c^a - r_{bc}\delta_d^a) = 0(3.3) \]

Since $M$ is manifold of class $R_1$, So by the Lemma 3.3 we have

\[ \frac{1}{2(n-1)}(r_{bd}\delta_c^a - r_{bc}\delta_d^a) = 0(3.4) \]

Contracting the equation (3.4) by the indices $a$ and $c$, we obtain

\[ \frac{1}{2(n-1)}(r_{bd}\delta_a^b - r_{ba}\delta_d^b) = 0 \]

Or equivalently,

\[ \frac{1}{2(n-1)}(n-1)r_{bd} = 0 \]

Therefore, $r_{bd} = 0$ and this complete the proof.

**Theorem 3.5.** Suppose that $M$ is flat $VG$-manifold with flat conformal curvature tensor, then $M$ is of zero scalar curvature tensor.

**Proof.** By using Lemma 3.2 we have

\[ W_{\bar{a}b\bar{c}d} = R_{\bar{a}b\bar{c}d} + \frac{2}{(n-1)}r_{[e}^a\delta_{d]}^b + \frac{k\delta_{ed}^{ab}}{2(n-1)(2n-1)} \]

(3.5)

Suppose that $M$ is flat $VG$-manifold with flat conformal curvature tensor. This means that the Riemannian and conformal curvature tensors are vanishing. Thus equation (3.5) becomes

\[ \frac{2}{(n-1)}(r_{c}^a\delta_c^b + r_{d}^b\delta_c^a - r_{c}^b\delta_c^b - r_{d}^a\delta_c^c) + \frac{k(\delta_{c}^a\delta_{d}^b - \delta_{c}^b\delta_{d}^a)}{2(n-1)(2n-1)} = 0 \]

(3.6)

Contracting (3.6) by the indexes $(b, d)$ and $(a, c)$ we get:

\[ \frac{2}{n-1}(r_{c}^a\delta_b^c + r_{d}^b\delta_a^d - r_{c}^b\delta_a^d - r_{d}^a\delta_c^c) + \frac{k(\delta_{c}^a\delta_{d}^b - \delta_{c}^b\delta_{d}^a)}{2(n-1)(2n-1)} = 0 \]

\[ 4r_{a}^a + \frac{nK}{2(2n-1)} = 0 \]

By using Lemma 2.5 we get

\[ \left(2 + \frac{n}{2(2n-1)}\right)K = 0 \]
Hence, $K = 0$

Therefore, $M$ is of zero scalar curvature tensor.

Similarly to the Lemma 3.3 we can construct the three special classes of $AH$-manifold depend on conformal curvature tensor, which are embodied in the following Lemma.

**Lemma 3.6.** In the adjoined $G$—structure space, an $AH$-manifold is manifold of class:

$W_1$ if and only if $W_{abcd} = W_{\bar{a}bcd} = W_{\bar{a}b\bar{c}d} = 0$,

$W_2$ if and only if $W_{abcd} = W_{\bar{a}b\bar{c}d} = 0$,

$W_3(WRK$—manifold) if and only if $W_{\bar{a}bcd} = 0$.

**Theorem 3.7.** Suppose that $M$ is $VG$-manifold of class $W_1$ and of flat Ricci tensor then $M$ is Kahler manifold.

**Proof.** In the adjoined $G$-structure space, the components of conformal curvature tensor can be written as follows

$$W_{\bar{a}bcd} = W(\varepsilon_{\bar{a}}, \varepsilon_{\bar{b}}, \varepsilon_{c}, \varepsilon_{d})$$

$$= W(\varepsilon_{\bar{a}}, \varepsilon_{\bar{b}}, J_{\varepsilon_c} J_{\varepsilon_d})$$

$$W(\varepsilon_{\bar{a}}, \varepsilon_{\bar{b}}, \sqrt{-1} \varepsilon_{c}, \sqrt{-1} \varepsilon_{d})$$

$$= (\sqrt{-1})(\sqrt{-1}) W(\varepsilon_{\bar{a}}, \varepsilon_{\bar{b}}, \varepsilon_{c}, \varepsilon_{d})$$

$$- W(\varepsilon_{\bar{a}}, \varepsilon_{\bar{b}}, \varepsilon_{c}, \varepsilon_{d}) = - W_{\bar{a}b\bar{c}d}$$

Thus,

$$2W_{\bar{a}bcd} = 0$$

Suppose that $M$ is $VG$-manifold of class $W_1$.

By using Lemmas 2.2 and 3.2, it follows that

$$-4B^{ab} B_{hcd} + 4a_{[c}^{[a} b^{b]d} + \frac{4}{n-1} r_{[c}^{[a} b^{b]d} + \frac{k \delta^{cd}_{cd}}{(n-1)(2n-1)} = 0(3.7)$$

Contracting (3.7) by indexes $(a, c)$ and $(b, d)$ we get:

$$-4B^{ab} B_{hab} + 4a_{[a}^{[a} b^{b]} + \frac{4}{n-1} r_{[a}^{[a} b^{b]} + \frac{2k \delta^{ab}}{2(n-1)(2n-1)} = 0$$

Or equivalently,

$$-4B^{ab} B_{hab} + 4n a^{a} + \frac{4n}{n-1} r_{a}^{a} + \frac{2n r_{a}^{a}}{(2n-1)} = 0$$

Since $M$ is manifold of flat Ricci tensor, then we get

$$-4B^{ab} B_{hab} + 4n a^{a} = 0(3.8)$$

Symmetrizing (3.8) by the indexes $(a, b)$, it follows that

$$4n a^{a} = 0$$

Thus, $a^{a} = 0$ (3.9)

Making use of the equations (3.9) and (3.8), it follows that
According to the Banaru’s classification we get that $M$ is Kahler manifold. ■

References


