The Generalized Taylor Expansion Method for Solving Some Types of Fractional Non-local Problems

Ahlam Jameel Khaleel and Hala Fouad Essa
Department of Mathematics, College of Science, Al-Nahrain University, Baghdad-Iraq.

Abstract
The aim of this paper is to prove the existence and the uniqueness of the solution for some types of fractional non-local problems, namely the non-linear non-local initial value problems for fractional Fredholm-Volterra integro-differential equations. Also, the generalized Taylor expansion method is used to solve the non-local initial value problem that consists of the linear fractional Fredholm-Volterra integro-differential equation together with the linear non-local initial condition with some illustrative examples.

Keywords: Non-Local Problems, Taylor Expansion Method, Fractional Fredholm-Volterra Integro-Differential Equations.

Introduction
The nonlocal conditions for the initial value problems appear when values of the function on the interval are connected to values inside the domain. Such problems are known as nonlocal problems, [1]. Many researchers studied the nonlocal problems, say [1] discussed the existence and uniqueness for the solutions of the nonlocal initial value problems for the non-linear ordinary differential equations, [8] used the finite difference method to solve special types of nonlocal problems for partial differential equations, [7] used the homotopy perturbation method to solve some types of the non-local initial value problems of fractional differential and integro-differential equations.

The fractional nonlocal Problems have been studied by several researchers such as [1,3,11] discussed the Nonlocal Cauchy problem for fractional evolution equations, [1] discussed the Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions, [7] discussed the nonlocal problems for fractional differential equations in Banach space, [1] discussed the Nonlinear fractional differential equations with nonlocal fractional integro-differential boundary conditions.

Existence and Uniquenesses of the Solutions for the Non-Local Initial Value Problems for Non-Linear Fractional Fredholm-Volterra Integro-Differential Equations

Recall that if \( u \) is an absolutely continuous function on \([a,b]\), the left and the right hand Caputo fractional derivative of \( u \) of order \( \alpha > 0 \), can be defined as:

\[
\begin{align*}
CD_a^\alpha u(x) &= \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{(x-t)^{(n-\alpha-1)}}{t^{\alpha-1}} u^{(n)}(t) \, dt, \\
& \quad a \leq x \leq b
\end{align*}
\]

and

\[
\begin{align*}
CD_b^\alpha u(x) &= \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{(t-x)^{(n-\alpha-1)}}{(t-x)^{\alpha-1}} u^{(n)}(t) \, dt, \\
& \quad a \leq x \leq b
\end{align*}
\]

respectively, where \( n-1 < \alpha \leq n \), \( n \) is a non-negative integer, [7].

In this section we shall discuss the existence of the unique solution for the non-linear non-local initial value problem that consists of the non-linear fractional Fredholm-Volterra integro-differential equation of order \( \alpha \):

\[
\begin{align*}
CD_a^\alpha u(x) &= f(x, u(x)) + \int_a^b k(x, y, u(y)) \, dy + \int_a^x l(x, y, u(y)) \, dy, \\
& \quad a \leq x \leq b
\end{align*}
\]

Together with the non-linear non-local initial condition:
where \( u \in C[a, b], \quad k: [a, b] \times [a, b] \times R \longrightarrow R \) and \( l: [a, b] \times [a, b] \times R \longrightarrow R \) are continuous functions, \( f: [a,b] \times R \longrightarrow R \), \( w: [a, b] \times R \longrightarrow R \) are continuous functions and \( C_{D^\alpha}^{\alpha+} \) is the left hand Caputo fractional derivative of \( u \) of order \( \alpha \), \( 0 < \alpha \leq 1 \). To do this we shall give the following theorem.

**Theorem (1.7):**

Consider the non-linear non-local initial value problem given by equations (2.1)-(2.2).

If the following conditions are satisfied:

1. \( f \) and \( w \) satisfy the Lipschitz condition with respect to the second argument with Lipschitz constants \( F \) and \( W \) respectively.
2. \( k \) and \( l \) satisfy Lipschitz condition with respect to the third argument with Lipschitz constants \( K \) and \( L \) respectively.

\[
\frac{F(b-a)}{\Gamma(\alpha+1)} + \left( K+L \right) \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+1)} < 1. 
\]

Then the non-linear non-local initial value problem given by equations (2.1)-(2.2) has a unique solution.

**Proof:**

It is known that \( C[a,b] \) is a Banach space with respect to the following norm:

\[
\|u\|_{C[a,b]} = \sup_{a \leq x \leq b} |u(x)|.
\]

It is easy to check that the non-local initial value problem given by equations (2.1)-(2.2) is equivalent to the non-linear integral equation:

\[
\begin{align*}
&\frac{1}{\Gamma(\alpha)} \int_a^b (x-y)^{\alpha-1} f(y,u(y)) dy + \\
&\frac{1}{\Gamma(\alpha)} \int_a^b (x-y)^{\alpha-1} \left[ \int_a^b k(y,s,u(s)) ds \right] dy + \\
&\frac{1}{\Gamma(\alpha)} \int_a^b (x-y)^{\alpha-1} \left[ \int_a^b l(y,s,u(s)) ds \right] dy
\end{align*}
\]

Let \( A \) be an operator that is defined by

\[
\begin{align*}
&\frac{1}{\Gamma(\alpha)} \int_a^b (x-y)^{\alpha-1} f(y,u(y)) dy + \\
&\frac{1}{\Gamma(\alpha)} \int_a^b (x-y)^{\alpha-1} \left[ \int_a^b k(y,s,u(s)) ds \right] dy + \\
&\frac{1}{\Gamma(\alpha)} \int_a^b (x-y)^{\alpha-1} \left[ \int_a^b l(y,s,u(s)) ds \right] dy
\end{align*}
\]

\[
\begin{align*}
&\frac{1}{\Gamma(\alpha)} \int_a^b (x-y)^{\alpha-1} f(y,u(y)) dy + \\
&\frac{1}{\Gamma(\alpha)} \int_a^b (x-y)^{\alpha-1} \left[ \int_a^b k(y,s,u(s)) ds \right] dy + \\
&\frac{1}{\Gamma(\alpha)} \int_a^b (x-y)^{\alpha-1} \left[ \int_a^b l(y,s,u(s)) ds \right] dy
\end{align*}
\]

\[
\begin{align*}
&\frac{1}{\Gamma(\alpha)} \int_a^b (x-y)^{\alpha-1} f(y,u(y)) dy + \\
&\frac{1}{\Gamma(\alpha)} \int_a^b (x-y)^{\alpha-1} \left[ \int_a^b k(y,s,u(s)) ds \right] dy + \\
&\frac{1}{\Gamma(\alpha)} \int_a^b (x-y)^{\alpha-1} \left[ \int_a^b l(y,s,u(s)) ds \right] dy
\end{align*}
\]

\[
\begin{align*}
&\frac{1}{\Gamma(\alpha)} \int_a^b (x-y)^{\alpha-1} f(y,u(y)) dy + \\
&\frac{1}{\Gamma(\alpha)} \int_a^b (x-y)^{\alpha-1} \left[ \int_a^b k(y,s,u(s)) ds \right] dy + \\
&\frac{1}{\Gamma(\alpha)} \int_a^b (x-y)^{\alpha-1} \left[ \int_a^b l(y,s,u(s)) ds \right] dy
\end{align*}
\]

\[
\begin{align*}
&\frac{1}{\Gamma(\alpha)} \int_a^b (x-y)^{\alpha-1} f(y,u(y)) dy + \\
&\frac{1}{\Gamma(\alpha)} \int_a^b (x-y)^{\alpha-1} \left[ \int_a^b k(y,s,u(s)) ds \right] dy + \\
&\frac{1}{\Gamma(\alpha)} \int_a^b (x-y)^{\alpha-1} \left[ \int_a^b l(y,s,u(s)) ds \right] dy
\end{align*}
\]
\[ u(x) = \sum_{i=0}^{N} \left( \frac{C_{D_{a^+}}^{\alpha}}{\Gamma(i \alpha + 1)} \right) u \frac{(x-a)^i}{i!} \]
where \( \alpha \leq 1 \), then

**Remarks, [4]:**

(1) For \( \alpha = 1 \), theorem (1.7) reduces to the classical Taylor formula.

(2) The generalized Taylor series for \( u \in C[a, b] \) takes the form:

\[ u(x) = \sum_{i=0}^{\infty} \left( \frac{C_{D_{a^+}}^{\alpha}}{\Gamma(i \alpha + 1)} \right) u \frac{(x-a)^i}{i!} \]

(3) Suppose that

\[ \left( \frac{C_{D_{a^+}}^{\alpha}}{\Gamma(i \alpha + 1)} \right) u \in C[a, b], \ i=0,1,\ldots,N+1 \]

and \( \cdot < \alpha \leq 1 \), then

\[ u(x) \equiv u_N(x) = \sum_{i=0}^{N} \left( \frac{C_{D_{a^+}}^{\alpha}}{\Gamma(i \alpha + 1)} \right) u \frac{(x-a)^i}{i!} \]

Furthermore, the error term \( R_N(x) \) has the form:

\[ R_N(x) = \left( \frac{C_{D_{a^+}}^{\alpha}}{\Gamma((N+1) \alpha + 1)} \right) u \frac{(x-a)^{N+1}}{(N+1)!} \]

where \( a \leq c \leq x \ \forall \ x \in (a, b) \).

**The Generalized Taylor Expansion Method for Solving Linear Fractional Fredholm-Volterra Integro-Differential Equations with Non-local Initial Condition**

In this section we will use the generalized Taylor expansion method to solve the non-local initial value problem that consists of the linear fractional Fredholm-Volterra integro-differential equation of order \( \alpha \) of the second kind:

\[ \frac{d^\alpha}{dx^\alpha} u(x) + \int_{a}^{b} K(x,y) u(y) dy = f(x), \quad a \leq x \leq b \]

\[ u(a) = \gamma \]

where \( \gamma \) is the non-local initial condition.
\[ C \frac{D^\alpha u(x)}{a^+} = g(x) + \]
\[ \lambda_1 \int_a^b k(x, y)u(y)dy + \lambda_2 \int_a^x l(x, y)u(y)dy, \]
\[ 0 < \alpha \leq 1 \] \hspace{1cm} (1.4)

Together with the linear non-local initial condition:
\[ u(a) = \mu_1 \int_a^b u(y)dy + \mu_2 \] \hspace{1cm} (1.5)
where \( g, u \in C[a, b] \),

\( k: [a, b] \times [a, b] \longrightarrow R, \ell: [a, b] \times [a, b] \longrightarrow R \) are continuous functions,
\( C \frac{D^\alpha u(x)}{a^+} \) is the left hand Caputo derivative
of \( u \) of order \( \alpha, \mu, \mu, \lambda, \lambda \), are known constants.

To do this, we assume that the solution \( u \)
of the non-local initial value problem given by
equations (1.4)-(1.5) can be approximated as a
generalized Taylor's formula:
\[ u(x) \approx u_N(x) = \sum_{i=0}^{N} \left( \frac{C \frac{D^\alpha u}{a^+}}{\Gamma(i\alpha + 1)} \right) (x-a)^i, \]
\[ a \leq x \leq b \] \hspace{1cm} (1.7)

By substituting equation (1.7) into
equations (1.4)-(1.5), one can have:
\[ C \frac{D^\alpha u(x)}{a^+} = g(x) + \]
\[ \sum_{i=0}^{N} \frac{1}{\Gamma(i\alpha + 1)} \left( \frac{C \frac{D^\alpha u}{a^+}}{i} \right) (a) + \lambda_1 \int_a^x l(x, y)(y-a)^i dy + \lambda_2 \int_a^b k(x, y)(y-a)^i dy \]
and
\[ u(a) = \mu_1 \int_a^b u(y)dy + \mu_2 \]
\[ = \mu_1 \sum_{i=0}^{N} \frac{(b-a)^{i\alpha+1}}{\Gamma(i\alpha+2)} \left( \frac{C \frac{D^\alpha u}{a^+}}{i} \right) (a) \]
\[ + \mu_2 \] \hspace{1cm} (1.8)

So
\[ C \frac{D^\alpha u(a)}{a^+} = g(a) + \]
\[ \sum_{i=0}^{N} \frac{1}{\Gamma(i\alpha + 1)} \left( \frac{C \frac{D^\alpha u}{a^+}}{i} \right) (a) + \lambda_1 \int_a^b k(x, y)(y-a)^i dy + \lambda_2 \int_a^x l(x, y)(y-a)^i dy + \mu_2 \]
\[ = \mu_1 \sum_{i=0}^{N} \frac{(b-a)^{i\alpha+1}}{\Gamma(i\alpha+2)} \left( \frac{C \frac{D^\alpha u}{a^+}}{i} \right) (a) + \mu_2 \]
and
\[ \left[ 1 - \mu_1 (b-a) \right] u(a) = \mu_2 \]
\[ \sum_{i=0}^{N} \frac{(b-a)^{i\alpha+1}}{\Gamma(i\alpha+2)} \left( \frac{C \frac{D^\alpha u}{a^+}}{i} \right) (a) = \mu_2 \] \hspace{1cm} (1.9)

Let
\[ a_{i,0} = \frac{-\lambda_1}{\Gamma(i\alpha+1)} \int_a^b k(a, y)(y-a)^i \]
\[ i=0,1,...,N, f_0 = 1 - \mu_1 (b-a) \]
\[ = \mu_1 \frac{(b-a)^{i\alpha+1}}{\Gamma(i\alpha+2)}, \quad i=1,2,...,N. \]

Then equations (1.7)-(1.9) become:
\[ \sum_{i=0}^{N} a_{i,0} \left( \frac{C \frac{D^\alpha u}{a^+}}{i} \right) (a) + \]
\[ i=1 \]
\[ (1 + a_{1,0}) \frac{C \frac{D^\alpha u}{a^+}}{a^+} (a) = g(a) \] \hspace{1cm} (1.10)

and
\[ \sum_{i=0}^{N} f_i \left( \frac{C \frac{D^\alpha u}{a^+}}{i} \right) (a) = \mu_2 \]
\[ \sum_{i=0}^{N} m_i (x) \left( \frac{C \frac{D^\alpha u}{a^+}}{i} \right) (a) \]
\[ = \int_a^x l(x, y)(y-a)^i dy, \quad i=0,1,...,N \]

Then
\[
\left( \begin{array}{c}
\left( C_{D_{a^+}} \alpha \right)^{j+1} u(x) \\
\left( C_{D_{a^+}} \alpha \right)^j g(x)
\end{array} \right) + \\
\sum_{i=0}^{N} \left( \begin{array}{c}
\lambda_1 \left( \begin{array}{c}
\left( C_{D_{a^+}} \alpha \right)^{j} m_i(x) \\
\left( C_{D_{a^+}} \alpha \right)^j p_i(x)
\end{array} \right)
\end{array} \right) \\
\frac{1}{\Gamma(i_{\alpha}+1)}
\right]
\]

where \( j=1,2,\ldots,N-1 \).

So,
\[
\left( \begin{array}{c}
\left( C_{D_{a^+}} \alpha \right)^{j+1} u(x) \\
\left( C_{D_{a^+}} \alpha \right)^j g(x)
\end{array} \right) + \\
\sum_{i=0}^{N} \lambda_1 \left( \begin{array}{c}
\left( C_{D_{a^+}} \alpha \right)^{j} m_i(x) \\
\left( C_{D_{a^+}} \alpha \right)^j p_i(x)
\end{array} \right) \\
\frac{1}{\Gamma(i_{\alpha}+1)}
\right]
\]

where \( j=1,2,\ldots,N-1 \).

Let
\[
a_{i,j} = -\frac{\lambda_1 \left( \begin{array}{c}
\left( C_{D_{a^+}} \alpha \right)^{j} m_i(x) \\
\left( C_{D_{a^+}} \alpha \right)^j p_i(x)
\end{array} \right) \left( C_{D_{a^+}} \alpha \right)^j u(x)}{\Gamma(i_{\alpha}+1)},
\]
where \( i=0,1,\ldots,N, j=1,2,\ldots,N-1 \).

Then equation (1.11) becomes
\[
\left( \begin{array}{c}
\left( C_{D_{a^+}} \alpha \right)^{j+1} u(x) \\
\left( C_{D_{a^+}} \alpha \right)^j g(x)
\end{array} \right) + \\
\sum_{i=0}^{N} a_{i,j} \left( \begin{array}{c}
\left( C_{D_{a^+}} \alpha \right)^{j} u(x)
\end{array} \right) \\
\left( \begin{array}{c}
\left( C_{D_{a^+}} \alpha \right)^j g(x), j=1,2,\ldots,N-1
\end{array} \right)
\]

............... (1.11)

Thus, by evaluating equation (1.11) at each \( j=1,2,\ldots,N-1 \) and by using equations (1.5)-(1.11), one can have the following linear system of \( N+1 \) equations with \( (N+1) \) unknowns
\[
\left\{ \left( C_{D_{a^+}} \alpha \right)^i u(x) \right\}_{i=0}^{N} = 0.
\]

By solving the above linear system of equations, one can get the values of
\[
\left\{ \left( C_{D_{a^+}} \alpha \right)^i u(x) \right\}_{i=0}^{N}.
\]

These values are substituted into equation (1.7) to get the approximated solution of the non-local initial value problem given by equations (1.4)-(1.9).

To illustrate this method, consider the following example:

**Example:**

Consider the nonlocal initial value problem that consists of the fractional linear Fredholm-Volterra integro-differential equation of order \( 1 \):
\[
\frac{d}{dx}^{\alpha} u(x) + \frac{d}{dx}^{\beta} u(x) = f(x),
\]
subject to initial conditions
\[
u(0) = 0, \quad \frac{d}{dx}^{\gamma} u(0) = 0.
\]

where \( \alpha, \beta, \gamma \) are fractional orders.
together with the nonlocal linear initial condition:

\[ u(0) = \frac{1}{2} \int_0^\infty u(y)dy - \frac{7}{2} \] ............................. (1.19)\]

We use the generalized Taylor expansion method to solve this fractional linear nonlocal initial value problem. To do this, let \( N = 1 \), then equation (1.15) takes the form:

\[ u(x) \equiv u_1(x) = u(0) + \frac{\left( \left( C_{D^{0+}} \frac{1}{2} \right) u \right)(0)}{\Gamma \left( \frac{3}{2} \right)} \sqrt{x}, \]

\[ 0 < x \leq 1 \] ............................. (1.16)

Then the system given by equation (1.15) takes the form:

\[
\begin{bmatrix}
-1 & -8 \\
1 & 5 \sqrt{\pi} - 4 \\
2 & 5 \sqrt{\pi}
\end{bmatrix}
\begin{bmatrix}
u(0) \\
\left( C_{D^{0+}} \frac{1}{2} \right) u \end{bmatrix}(0)
\]

= \begin{bmatrix}
7/2 \\
3/2
\end{bmatrix}

which has the solution:

\[ u(0) = \frac{3(35 \sqrt{\pi} + 12)}{2(15 \sqrt{\pi} + 8)} \approx 3.21087 \]

and

\[ \left( C_{D^{0+}} \frac{1}{2} \right) u \right)(0) = 1 - \frac{2}{4 - 15 \sqrt{\pi} + 8} \approx 0.192174. \]

By substituting these values into equation (1.15) one can have:

\[ u(x) \equiv u_1(x) = 3.21087 + 0.216846 \sqrt{x}, \]

\[ 0 \leq x \leq 1 \]

By substituting this approximated solution into equation (1.15) one can have:
\[ u(x) = u_2(x) = 3.21087 + 0.216846x, \quad 0 \leq x \leq 1. \]

Since \( u_2(x) = u_1(x) \), we must increase the value of \( N \). By continuing in this manner one can get for \( N = \gamma \), equation (\( \gamma, \gamma \)) takes the form:

\[ u(x) \equiv u_6(x) \]

\[ = u(0) + \sum_{r=1}^{\gamma} \frac{1}{\Gamma \left( \frac{3}{2} \right)} x^{\frac{1}{2}} + \sum_{r=1}^{\gamma} \frac{1}{\Gamma \left( \frac{3}{2} \right)} x^{\frac{1}{2}} \]

\[ + \sum_{r=1}^{\gamma} \frac{1}{\Gamma \left( \frac{5}{2} \right)} x^{\frac{3}{2}} + \sum_{r=1}^{\gamma} \frac{1}{\Gamma \left( \frac{5}{2} \right)} x^{\frac{3}{2}} \]

\[ + \sum_{r=1}^{\gamma} \frac{1}{\Gamma \left( \frac{7}{2} \right)} x^{\frac{5}{2}} + \sum_{r=1}^{\gamma} \frac{1}{\Gamma \left( \frac{7}{2} \right)} x^{\frac{5}{2}} \]

\[ \leq x \leq 1 \]

\[ \sum_{i=1}^{5} \frac{1}{\Gamma \left( \frac{5}{2} \right)} x^{\frac{5}{2}} \]

Then the system given by equation (\( \gamma, \gamma \)) takes the form:

\[ \begin{pmatrix} -1 & -8 & -1 & -16 & -1 & -32 & -1 \\ -1 & 3 \pi & -1 & -1 & -16 & -1 & -1 \\ -1 & 5 \pi & -1 & -1 & -16 & -1 & -1 \\ 2 & 3 \pi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \]

which has the solution:

\[ u(0) = 1, \]

\[ \begin{pmatrix} \frac{1}{2} \end{pmatrix} u \end{pmatrix} (0) = 0, \quad i = 1, 2, 3, 4, 5 \]

and

\[ \begin{pmatrix} \frac{1}{2} \end{pmatrix} u \end{pmatrix} (0) = 30. \]

By substituting these values into equation (\( \gamma, \gamma \)) one can have:

\[ u(x) \equiv u_6(x) = 1 + 5x^3, \quad 0 \leq x \leq 1 \]

By substituting this approximated solution into equations (\( \gamma, \gamma \))-(\( \gamma, \gamma \)) one can get:
\[
C_{D}^{1/2} 0\frac{u_6(x)}{x} + \frac{3}{2} + 25x^2 - \frac{16}{\sqrt{\pi}} x^\frac{5}{2} + 23x^5
- \int_0^1 (x^2 + y)u_6(y)dy - \frac{x}{3}(3x + 2y)u_6(y)dy = 0
\]

and

\[
u_6(0) = \frac{1}{2} u_6(y)dy - \frac{7}{2}.
\]

Therefore \(u_6\) is the exact solution of the linear nonlocal problem given by equations (1.14)-(1.15).

Conclusions

From this work, one can conclude that the following aspects:

1. The existence and the uniqueness of the solution for the non-linear non-local initial value problem is a generalization of the existence and the uniqueness of the solution for the linear local initial value problem.

2. The generalized Taylor expansion method like the classical Taylor expansion method gave more accurate results as \(N\) increases.

References


