The Hypersolvable Complex Reflection Arrangement $A(G_{29})$

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ABSTRACT

The purpose of this study is to analyze the complex reflection arrangements $A(G_{29})$. During our study work we succeed to answering the open problem suggested by Jambu and Papadima in “Jambu M., Papadima S., [1]” (is there an arrangement $A$, hypersolvable and free, such that $L(A) > r(A)$?), when we prove that $A(G_{29})$ is a rank four hypersolvable arrangement with a composition series of length $L(A(G_{29})) = 25$, and $A(G_{29})$ is not supersolvable arrangement. We compute the Poincare polynomials of $A(G_{29})$. Also we prove that the action of $G_{29}$ on $A(G_{29})$ preserves the hypersolvability conditions.

INTRODUCTION

Let $V \cong C^r$ and choose coordinates for $V^*$ s.t. we can identify the symmetric algebra $S = S(V^*)$ with the polynomial ring $C[Z_1,...,Z_r]$. A hyperplane in $V$ is a codimension one affine space in $V$. A hyperplane arrangement in $V$ is a finite collection of hyperplanes denoted by $A$ [2]. When the hyperplane of an arrangement contain the origin we say the arrangement is central.

For each $H \in A$ choose a linear polynomial $\alpha_H \in S$ s.t $H = \text{Ker} \alpha_H$. Let $Q = \prod_{H \in A} \alpha_H$ denote the defining polynomial of the arrangement $A$.

Throughout this paper we use the following notations:-

Let $L_A$ be the intersection lattice of $A$ which is the set of all intersections of elements from $A$ with the order being reverse inclusion i.e. $X \leq Y \iff Y \subseteq X$, for each $X, Y \in L_A$.

§2: HYPERSOLVABLE & SUPERSOLVABLE ARRANGEMENTS

The next definitions are standard for lattices in general, see [3,4,5] we will use them for central arrangements.

Let $A = \{H_1, H_2, ..., H_\ell\}$ be a central arrangement in the complex vector space $V$. Denote also by $\mathscr{A}^p = \{\alpha_1, \alpha_2, ..., \alpha_\ell\} \subset P(V^*)$ its set of defining equations (forms) viewed as points in the dual projective space,
The Hypersolvable Complex Reflection Arrangement A(G29)

where \( H_i = \ker \alpha_i, \ i = 1, 2, \ldots, \ell \). It simplifies notation to assume that \( L = L_A \) and \( A \) is essential. Let \( \text{rk}(A) = n \) and \( T = T(A) \).

**Lemma (2.1)** [6]: An element \( X \in L \) is modular if and only if \( X + Y \in L, \ \forall \ Y \in L \).

**Definition (2.2)**[4]: Let \( A \) be an arrangement with \( \text{rk}(A) = n \), and \( L \) be its lattice, then we call \( A \) supersolvable if \( L \) has a maximal chain of modular elements 
\[ V = X_0 < X_1 < \ldots < X_n = T \]

**Definition (2.3)**[7]: Let \( B \subset A \) be a proper non-empty sub-arrangement of \( A \) and set \( B^c = A - B \). We say that \( (A, B) \) is a solvable extension if the following conditions are satisfied:
1. The extension is closed: If no point \( a \in B^c \) sits on a projective line determined by \( \alpha, \beta \in B \), i.e. \( \text{rk}\{\alpha, \beta, a\} = 3 \).
2. The extension is complete: If for every \( a, b \in B^c \) with \( a \neq b \), there exists a point \( \gamma \in B \), on the line passing through \( a \) and \( b \), i.e., \( \text{rk}\{a, b, \gamma\} = 2 \).
3. For every distinct points \( a, b, c \in B^c \), the three points \( \alpha, \beta, \gamma \) determined by \( (a, b), (b, c) \) and \( (a, c) \) respectively are either equal or collinear.

**Definition (2.4)** [7]: The arrangement \( A \) is called hypersolvable if it has a hypersolvable composition series. i.e., an ascending chain of sub-arrangements
\[ A_1 \subset A_2 \subset \ldots \subset A_i \subset A_{i+1} \subset \ldots \subset A_k = A, \ \text{where} \ \text{rk}(A_1) = 1, \ \text{and each} \ (A_i, A_{i+1}) \ \text{is a solvable extension}. \ \text{The positive integer} \ k \geq 2 \ \text{is called the length of the composition series}, \ \text{which is denoted by} \ L(A), \ \text{and depends only on} \ A. \ \text{i.e.} \ L(A) = k.

**Theorem (2.5)** [7]: Let \( A \) be a hypersolvable. Then \( A \) is supersolvable if and only if \( L(A) = \text{rk}(A) \).

Notice that, theorem (2.5) provides a very simple supersolvability test.

§3: The Complex Reflection Arrangement A (G29)
The classification of finite reflection groups contains three infinite families and (34) exceptional groups labeled \( G_4 - G_{37} \) [8]. Our interest group is \( G_{29} \).

**The Complex Reflection Group G29 (3.1):**
\( G_{29} \) is a subgroup of PGL (4,C). It give rise to a complex reflection group \( G \subset U(\mathbb{C}^4) \cong U(4,\mathbb{C}) \subset GL(4,\mathbb{C}), \) of order 7680. It is generated by reflections.
All the reflection of \( G_{29} \) are of order 2, but \( G_{29} \) is not the complexification of a real group. It is well – generated irreducible – reflection group [2].

The corresponding reflection arrangement has "40" hyperplanes and defined by [9]:
\[ Q(A_{(G_{29})}) = xyzw(x-y)(x-z)(x-w)(y-z)(y-w)(z-w)(x+y)(x+z)(x+w)(y+z)(y+w)(z+w) \]
\[(x-y+iz+ iw) \ (x-y+iz-iw) \ (x-y-iz+ iw) \ (x+y+iz+ iw) \ (x+y-iz-iw)
\]
\[(x+y-iz+ iw) \ (x+y+iz-w) \ (x-iy+iz+w) \ (x-iy-iz+w)\]
\[(x-iy-iz-w) \ (x+iy-iz+w) \ (x+iy+iz-w) \ (x+iy+z+ iw)\]
\[(x-iy+z+ iw) \ (x+iy+z+iw) \ (x+iy-z+iw) \ (x+iy+z-iw)\]

The Lattice of \(A(G_{29})\) (3.2):

Recall that the hyperplanes of an arrangement \(A\) in \(C^4\) are represented as usual by lines in the projective plane \(P(C^4)\). The hyperplanes of the reflection arrangement \(A(G_{29})\) represented by \(H_i, \ i = 1, \ldots, 40\) as follows:

<table>
<thead>
<tr>
<th>(H_1)</th>
<th>(H_{11})</th>
<th>(H_{21})</th>
<th>(H_{31})</th>
</tr>
</thead>
<tbody>
<tr>
<td>: (x = 0)</td>
<td>: (x+y = 0)</td>
<td>: (x+y+iz+iw = 0)</td>
<td>: (x+y+iz+w = 0)</td>
</tr>
<tr>
<td>(H_2)</td>
<td>(H_{12})</td>
<td>(H_{22})</td>
<td>(H_{32})</td>
</tr>
<tr>
<td>: (y = 0)</td>
<td>: (x+z = 0)</td>
<td>: (x+y-iz-iw = 0)</td>
<td>: (x+y+iz-w = 0)</td>
</tr>
<tr>
<td>(H_3)</td>
<td>(H_{13})</td>
<td>(H_{23})</td>
<td>(H_{33})</td>
</tr>
<tr>
<td>: (z = 0)</td>
<td>: (x+w = 0)</td>
<td>: (x+y-iz+iw = 0)</td>
<td>: (x-iy+z+iw = 0)</td>
</tr>
<tr>
<td>(H_4)</td>
<td>(H_{14})</td>
<td>(H_{24})</td>
<td>(H_{34})</td>
</tr>
<tr>
<td>: (w = 0)</td>
<td>: (y+z = 0)</td>
<td>: (x+y+iz-iw = 0)</td>
<td>: (x-iy+z-iw = 0)</td>
</tr>
<tr>
<td>(H_5)</td>
<td>(H_{15})</td>
<td>(H_{25})</td>
<td>(H_{35})</td>
</tr>
<tr>
<td>: (x-y = 0)</td>
<td>: (y+w = 0)</td>
<td>: (x+y+iz+w = 0)</td>
<td>: (x-iy+z-w = 0)</td>
</tr>
<tr>
<td>(H_6)</td>
<td>(H_{16})</td>
<td>(H_{26})</td>
<td>(H_{36})</td>
</tr>
<tr>
<td>: (x-z = 0)</td>
<td>: (z+w = 0)</td>
<td>: (x-y+iz+ w = 0)</td>
<td>: (x-iy-z+w = 0)</td>
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<tr>
<td>(H_7)</td>
<td>(H_{17})</td>
<td>(H_{27})</td>
<td>(H_{37})</td>
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<tr>
<td>: (x-w = 0)</td>
<td>: (x-y+iz+ w = 0)</td>
<td>: (x-iy-z+ w = 0)</td>
<td>: (x+iy+z+ w = 0)</td>
</tr>
<tr>
<td>(H_8)</td>
<td>(H_{18})</td>
<td>(H_{28})</td>
<td>(H_{38})</td>
</tr>
<tr>
<td>: (y-z = 0)</td>
<td>: (x+y+iz- w = 0)</td>
<td>: (x-iy-z- w = 0)</td>
<td>: (x+iy-z-w = 0)</td>
</tr>
<tr>
<td>(H_9)</td>
<td>(H_{19})</td>
<td>(H_{29})</td>
<td>(H_{39})</td>
</tr>
<tr>
<td>: (y-w = 0)</td>
<td>: (x+y+iz- w = 0)</td>
<td>: (x+y+iz+w = 0)</td>
<td>: (x+y+iz+w = 0)</td>
</tr>
<tr>
<td>(H_{10})</td>
<td>(H_{20})</td>
<td>(H_{30})</td>
<td>(H_{40})</td>
</tr>
<tr>
<td>: (z-w = 0)</td>
<td>: (x+y+iz+ w = 0)</td>
<td>: (x+y+iz-w = 0)</td>
<td>: (x+y+z-w = 0)</td>
</tr>
</tbody>
</table>

In the case of \(A(G_{29})\), the projective (40) lines meet in "310" points. Each of the first "120" points is contained in "7" lines, so we refer to them as sevenple points.

Each of the second "160" points is contained in "3" lines, so we refer to them as triple points. The last "30" points is contained in "10" lines, so we refer to them as tenple points. Thus each projective line contains "24" points: "6" tenple points, "6" sevenple points and "12" triple points.

Therefore the lattice of \(A(G_{29})\) consists of "732" elements which are:

1- \(C^4\) as the minimal element.
2- The set of atoms (40 hyperplanes) as the rank one elements.
3- The rank two elements consists of "310" elements.
4- The rank three elements consists of "380" elements.
5- The maximal element \(T = \bigcap_{i=1}^{40} H_i = \{0\}\).

**Remark (3.3):**

We do not refer to the lattice of \(A(G_{29})\) in details. Since it contains a big number of intersection points (732 elements). But we find the intersection points of \(H_1, H_2, H_3\) and \(H_4\) for our interest:

\(H_1 \cap H_2 \cap H_{25} \cap H_{28} \cap H_{30} \cap H_{31} \cap H_{34} \cap H_{36} \cap H_{39} \cap H_{40} \), \(x=y=0, \ iz+w=0\).
H₁ ∩ H₂ ∩ H₂₆ ∩ H₂₇ ∩ H₂₉ ∩ H₃₂ ∩ H₃₅ ∩ H₃₇ ∩ H₃₈ : x = y = 0, iz - w = 0.
H₁ ∩ H₃ ∩ H₁₇ ∩ H₂₀ ∩ H₂₂ ∩ H₂₄ ∩ H₂₆ ∩ H₂₈ ∩ H₂₉ ∩ H₃₁ : x = z = 0, -y + iw = 0.
H₁ ∩ H₃ ∩ H₁₈ ∩ H₁₉ ∩ H₂₁ ∩ H₂₃ ∩ H₂₅ ∩ H₂₇ ∩ H₃₀ ∩ H₃₂ : x = z = 0, iy - w = 0.
H₁ ∩ H₄ ∩ H₁₇ ∩ H₁₈ ∩ H₂₀ ∩ H₂₂ ∩ H₂₃ ∩ H₂₅ ∩ H₂₇ ∩ H₃₆ ∩ H₃₇ ∩ H₄₀ : x = w = 0, iy + z = 0.
H₁ ∩ H₄ ∩ H₁₉ ∩ H₂₀ ∩ H₂₁ ∩ H₂₃ ∩ H₂₄ ∩ H₃₃ ∩ H₃₄ ∩ H₃₈ ∩ H₃₉ : x = w = 0, iy - z = 0.
H₂ ∩ H₃ ∩ H₁₇ ∩ H₂₀ ∩ H₂₁ ∩ H₂₃ ∩ H₂₅ ∩ H₂₆ ∩ H₃₁ ∩ H₃₂ : y = w = 0, x + iz = 0.
H₁ ∩ H₂ ∩ H₁₀ ∩ H₁₈ ∩ H₂₀ ∩ H₂₃ ∩ H₂₅ : x = y = 0, z - w = 0.
H₁ ∩ H₂ ∩ H₁₆ ∩ H₁₇ ∩ H₁₉ ∩ H₂₁ ∩ H₂₂ : x = y = 0, z + w = 0.
H₁ ∩ H₃ ∩ H₅ ∩ H₃₃ ∩ H₃₆ ∩ H₃₈ ∩ H₄₀ : x = z = 0, y - w = 0.
H₁ ∩ H₃ ∩ H₁₅ ∩ H₂₇ ∩ H₂₈ ∩ H₃₁ ∩ H₃₂ : x = z = 0, y + w = 0.
H₁ ∩ H₄ ∩ H₂ ∩ H₂₅ ∩ H₂₆ ∩ H₂₉ ∩ H₃₀ : x = w = 0, y - z = 0.
H₁ ∩ H₄ ∩ H₁₄ ∩ H₂₇ ∩ H₂₈ ∩ H₃₁ ∩ H₃₂ : x = w = 0, y + z = 0.
H₁ ∩ H₁₇ ∩ H₂₂ : x = 0, y - iz - iw = 0.
H₁ ∩ H₁₈ ∩ H₂₃ : x = 0, y - iz + iw = 0.
H₁ ∩ H₁₉ ∩ H₂₁ : x = 0, y - iz - iw = 0.
H₁ ∩ H₂₀ ∩ H₂₄ : x = 0, y + iz - iw = 0.
H₁ ∩ H₂₅ ∩ H₃₀ : x = 0, iy - iz - w = 0.
H₁ ∩ H₂₆ ∩ H₂₉ : x = 0, iy - iz + w = 0.
H₁ ∩ H₂₇ ∩ H₃₂ : x = 0, iy + iz - w = 0.
H₁ ∩ H₂₈ ∩ H₃₁ : x = 0, iy + iz + w = 0.
H₁ ∩ H₃₃ ∩ H₃₈ : x = 0, iy - z + iw = 0.
H₁ ∩ H₃₄ ∩ H₃₉ : y = 0, iy - z + iw = 0.
H₂ ∩ H₁₇ ∩ H₂₁ : y = 0, x + iz - iw = 0.
H₂ ∩ H₁₈ ∩ H₂₄ : y = 0, x + iz + iw = 0.
H₂ ∩ H₁₉ ∩ H₂₂ : y = 0, x - iz - iw = 0.
H₂ ∩ H₂₀ ∩ H₂₃ : y = 0, x - iz + iw = 0.
H₂ ∩ H₂₅ ∩ H₃₁ : y = 0, x + iz + w = 0.
H₂ ∩ H₂₆ ∩ H₃₂ : y = 0, x + iz - w = 0.
H₂ ∩ H₂₇ ∩ H₂₉ : y = 0, x - iz + w = 0.
H₂ ∩ H₂₈ ∩ H₃₀ : y = 0, x - iz - w = 0.
H₂ ∩ H₃₃ ∩ H₃₇ : y = 0, x + z + iw = 0.
H₂ ∩ H₃₄ ∩ H₄₀ : y = 0, x + z - iw = 0.
H₂ ∩ H₃₅ ∩ H₃₈ : y = 0, x - z - iw = 0.
H₂ ∩ H₃₆ ∩ H₃₉ : y = 0, x - z + iw = 0.
H₃ ∩ H₁₇ ∩ H₂₀ : z = 0, x - y + iw = 0.
H₃ ∩ H₁₈ ∩ H₁₉ : z = 0, x - y - iw = 0.
H₃ ∩ H₂₁ ∩ H₂₃ : z = 0, x + y + iw = 0.
H₃ ∩ H₂₂ ∩ H₂₄ : z = 0, x + y - iw = 0.
H₃ ∩ H₂₅ ∩ H₂₇ : z = 0, x - iy + w = 0.
Finally, to compute the Poincare polynomial of \( A(G_{29}) \), note that if \( X \in L_A \) s.t. \( \text{rk}(X) = 1 \), then \( \mu(X) = -1 \), if \( \text{rk}(X) = 2 \), and \( X \) is in three planes then \( \mu(X) = 2 \), and if \( \text{rk}(X) = 2 \), s.t. \( X \) is in seven planes then \( \mu(X) = 6 \), if \( \text{rk}(X) = 2 \), s.t. \( X \) is in ten planes then \( \mu(X) = 9 \). This allows calculation of \( \mu(\emptyset) \).

Thus \( \pi(A, t) = 1+40t+331t^2+2470t^3+1989t^4 \).

**The Orbits of \( G_{29} \) on \( L_A(G_{29}) \) (3.4) [9].**

The complex reflection group \( G_{29} \) has eleven orbits on \( L_A(G_{29}) \) which are:

1- \( C^4 \) has fixer the identity group \( A_0 \).
2- "40" hyperplanes form a single orbit with fixer \( A_1 \).
3- "120" lines form an orbit with fixer \( A_1 x A_1 \).
4- "160" lines form an orbit with fixer \( A_2 \).
5- "30" lines form an orbit with fixer \( B_2 \).
6- "160" lines form an orbit with fixer \( A_1 x A_2 \).
7- "160" lines form two orbits each orbit consists of "80" lines with fixers \( A_3', A_3'' \).
8- "40" lines form an orbit with fixer \( B_3 \).
9- "20" lines form an orbit with fixer \( G(4,4,3) \).
10- The origin is an orbit with fixer \( G_{29} \).

**Proposition (3.5):**
The reflection arrangement $A(G_{29})$ is a hypersolvable arrangement with a composition series of length "25".

**Proof:**

We have to find a composition series (at least one) using definition (2.4), this series of length "25" bigen with one hyperplane say $H_i$, $1 \leq i \leq 40$, as the step one of this series, i.e. $\mathcal{A}_i = \{H_i\}$, each hyperplane $H_i$, $(1 \leq i \leq 40)$ as a line in $\mathbb{P}(\mathbb{C}^d)$ contains "24" intersecting points, "6" tenple points "6" sevenple points and "12" triple points.

For step "2" of the series $A_2$ consists of ten hyperplanes (including $H_i$) intersect in one point say $P_1$, for step "3" $A_3$ consists of $A_2$ and nine hyperplanes intersect $H_i$ in one point say $P_2$, for step "4" $A_4$ consists of $A_3$ and nine hyperplanes intersect $H_i$ in one point say $P_3$, for step "5" $A_5$ consists of $A_4$ and nine hyperplanes intersect $H_i$ in one point say $P_4$, for step "6" $A_6$ consists of $A_5$ and nine hyperplanes intersect $H_i$ in one point say $P_5$, for step "7" $A_7$ consists of $A_6$ and nine hyperplanes intersect $H_i$ in one point say $P_6$, for step "8" $A_8$ consists of $A_7$ and six hyperplanes intersect $H_i$ in one point say $q_1$, for step "9" $A_9$ consists of $A_8$ and six hyperplanes intersect $H_i$ in one point say $q_2$, for step "10" $A_{10}$ consists of $A_9$ and six hyperplanes intersect $H_i$ in one point say $q_3$, for step "11" $A_{11}$ consists of $A_{10}$ and six hyperplanes intersect $H_i$ in one point say $q_4$, for step "12" $A_{12}$ consists of $A_{11}$ and six hyperplanes intersect $H_i$ in one point say $q_5$, for step "13" $A_{13}$ consists of $A_{12}$ and six hyperplanes intersect $H_i$ in one point say $q_6$, for step "14" $A_{14}$ consists of $A_{13}$ and two hyperplanes intersect $H_i$ in one point say $t_1$, for step "15" $A_{15}$ consists of $A_{14}$ and two hyperplanes intersect $H_i$ in one point say $t_2$, for step "16" $A_{16}$ consists of $A_{15}$ and two hyperplanes intersect $H_i$ in one point say $t_3$, for step "17" $A_{17}$ consists of $A_{16}$ and two hyperplanes intersect $H_i$ in one point say $t_4$ for step "18" $A_{18}$ consists of $A_{17}$ and two hyperplanes intersect $H_i$ in one point say $t_5$ for step "19" $A_{19}$ consists of $A_{18}$ and two hyperplanes intersect $H_i$ in one point say $t_6$ for step "20" $A_{20}$ consists of $A_{19}$ and two hyperplanes intersect $H_i$ in one point say $t_7$ for step "21" $A_{21}$ consists of $A_{20}$ and two hyperplanes intersect $H_i$ in one point say $t_8$ for step "22" $A_{22}$ consists of $A_{21}$ and two hyperplanes intersect $H_i$ in one point say $t_9$ for step "23" $A_{23}$ consists of $A_{22}$ and two hyperplanes intersect $H_i$ in one point say $t_{10}$ for step "24" $A_{24}$ consists of $A_{23}$ and two
hyperplanes intersect \( H_i \) in one point say \( t_{11} \) for step "25" \( A_{25} \) consists of \( A_{24} \) and two hyperplanes intersect \( H_i \) in one point i.e. \( (A_{25} = A_{(G_{29})}) \)

The resulting composition series is a hypersolvable composition series since \( A_i \) is solvable in \( A_{i+1} \), \( 1 \leq i \leq 24 \) , also for any distinct hyperplanes \( H_{i_j}, H_{i_k}, H_{i_0} \in A_i^c \), we have \( \gamma_{H_{i_j}, H_{i_k}} = \gamma_{H_{i_k}, H_{i_0}} = \gamma_{H_{i_j}, H_{i_0}} = H_{i_j} \). Therefore \( A(G_{29}) \) is a hypersolvable arrangement.

**Corollary (3.6):**
The reflection arrangement \( A_{(G_{29})} \) is not supersolvable arrangement.

**Proof:**
From the proof of proposition (3.5) we have \( A(G_{29}) \) is a rank four hypersolvable arrangement with a composition series of length equal to "25", so by Theorem (2.5) \( A(G_{29}) \) is not a supersolvable arrangement.

**Proposition (3.7):**
The reflection group \( G_{29} \) preserves the hypersolvability condition on \( A_{(G_{29})} \).

**Proof:**
We proved in proposition(3.5) that \( A(G_{29}) \) is a hypersolvable arrangement, i.e. \( A(G_{29}) \) has a hypersolvable composition series:
\[
A_1 \subset A_2 \subset \ldots \subset A_{25} = A(G_{29}) \quad (1)
\]

With \( \text{rk}(A_1) = 1 \) and \( \text{rk}(A_{25}) = 4 \) s.t. \( (A_i, A_{i+1}) \) is solvable extension , \( 1 \leq i \leq 24 \).

We know that the complex reflection group \( G_{29} \) acts on \( A(G_{29}) \) by permutation and the induced action on \( \mathcal{L}(A(G_{29})) \) is order preserving.

Let \( A_1 \) be the Coxeter group which is isomorphic to the symmetric group \( S_2 \).

Let \( \sigma \) be the transposition \( (12) \in S_2 \), and let \( H_{i_1}, H_{i_2} \in A(G_{29}), 1 \leq i \leq 40 \).

Then we have the following cases:

1- **Case One:**
If \( H_{i_1} \in A_1 \) and \( H_{i_2} \in A_p \), \( p = 2, 3, \ldots, 25 \). Then \( \sigma \) acts on the composition series (1) and the result is also a composition series
\[
A_{\sigma(1)} \subset A_{\sigma(2)} \subset \ldots \subset A_{\sigma(24)} = A(G_{29}) .
\]

2- **Case Two:**
If \( H_{i_1}, H_{i_2} \in A_p \), \( p = 2, 3, \ldots, 25 \). Then \( \sigma \) acts on (1) and the result is a composition series,
\[
A_1 \subset A_{\sigma(2)} \subset \ldots \subset A_{\sigma(25)} = A(G_{29}) .
\]

3- **The Last Case:**
when $H_{i_1} \in \mathcal{A}_p$ and $H_{i_2} \in \mathcal{A}_q$, $(2 \leq P < q \leq 25)$ and $\sigma$ acts on (1), then the result is also a composition series, $A_1 \subset A_{\sigma(2)} \subset \ldots \subset A_{\sigma(25)} = A(G_{29})$.

Thus we proved the existence of the composition series.

Now, we have to prove that the action of $\sigma$ on (1) preserves the solvability.

i.e. to prove that $\mathcal{A}_{\sigma(j)}$ is solvable in $\mathcal{A}_{(\sigma_{j+1})}$, $j = 1, 2, \ldots, 24$

In case one we show that $H_{i_2} \in \mathcal{A}_{\sigma(1)}$ and $H_{i_1} \in \mathcal{A}_{\sigma(2)}$, i.e. $H_{i_1} \in \mathcal{A}_{\sigma(1)}^c$, and for any $H_{i_{o'}} \in \mathcal{A}_{\sigma(1)}^c$, we have $\text{rk}\{H_{i_1}, H_{i_2}, H_{i_{o'}}\} = 2$, $3 \leq \omega \leq n_o$, where $n_o = |A_2|$. Thus $\mathcal{A}_{\sigma(1)}$ is complete in $\mathcal{A}_{\sigma(2)}$.

In case two we have the same thing as case one. In case three when $H_{i_1} \in \mathcal{A}_{\sigma(q)}$ and $H_{i_2} \in \mathcal{A}_{\sigma(p)}$, $(2 \leq P < q \leq 25)$ i.e. $H_{i_1} \in \mathcal{A}_{\sigma(p)}^c$ and for any $H_{i_{o'}} \in \mathcal{A}_{\sigma(p)}^c$ we have $\text{rk}\{H_{i_1}, H_{i_2}, H_{i_{o'}}\} = 2$, $(2 \leq \omega' \leq m_1 = |A_q|)$.

Clear that $\sigma$ preserves the closedness condition in each case. Therefore $\sigma$ preserves the solvability condition. i.e. $A(G_{29})$ is a hypersolvable under the action of $G_{29}$ on it.

REFERENCES