On The Stability Of Fixed Points In Topological Spaces

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Abstract

In this paper we introduced new concepts of stability: strongly stable, c-stable, and c-strongly stable. We discussed the stability of fixed points with respect to each of a topology and a base for that topology. We obtained that if a fixed point is stable (strongly stable, c-stable, strongly c-stable respectively) with respect to a base for a topology, then it is stable (strongly stable, c-stable, strongly c-stable respectively) with respect to that topology.

Keywords: Fixed points, Stable fixed point, Orbit, Dynamical system, and Convergent sequences.

Introduction

A strong concept of stability for dynamical system were first formulated by Russian scientist N.E. Zhukovskii (1847-1921) [3]. He introduced in 1882 a strong orbital stability concept which is based on a reparametrisation of the time variable [3]. In 1892, the Russian scientist A.M. Lyapunov, defined the concept of stability in his PhD thesis "A general task about the stability of motion" [3]. The work of Lyapunov became famous in Russia and after that in the West. His thesis was translated into French in 1907, and it is reprinted in Russian in 1950 [3]. An English translation and biography has been published in 1992 [2]. The notation of Lyapunov differs a little from the modern notation. Lyapunov introduces \( n \) quantities \( F_i \) which are functions of the \( k \) trajectories \( f_j(t) \) of the positions \( q_j \) starting from the unperturbed initial condition \( q_{j0} \). The quantities \( Q_i \) denote these functions for the perturbed trajectories due to perturbations \( \varepsilon_j \) on the initial position and \( \varepsilon_j^i \) on the initial velocity [3].

1. Preliminaries

1.1 Definition [4]

Let \( (X, \tau) \) be a topological space. A base \( B_\tau \) for topology \( \tau \) is subcollection of \( \tau \), such that every nonempty open set in \( X \) is a union of some members of \( B_\tau \), i.e.:

i) \( B_\tau \subseteq \tau \), and

ii) \( \forall W \in \tau, W \neq \emptyset \Rightarrow W = \bigcup_{a \in B_\tau} B_a \)

1.2 Example

Consider that the topological space \( (R, \tau_u) \):
\( \tau_u \) is the usual topology on the real numbers \( R \). \( B_{\tau_u} = \{(a, b) \in \tau_u; a, b \in R\} \) is a base for \( \tau_u \).

1.3 Definition [5]

Let \((X, \tau)\) be a topological space, \((a_n)_{n \in \mathbb{Z}^+}\) be a sequence in \(X\). Then \((a_n)\) is said to be converges to \(x_0 \in X\), or \(x_0\) is limit of \((a_n)_{n \in \mathbb{Z}^+}\) \((a_n \to x_0)\) iff for every open set \(H\) of \(X\) containing \(x_0\) containing all but finitely many elements of \((a_n)\). Hence \(a_n \to 0\).

1.4 Example

Consider the topological space \((R, \tau_u)\), and \((a_n) = \left(\frac{1}{n^2}, n \in \mathbb{Z}^+\right)\).
Let \(G = (a, b) \in \tau\) where \(0 \in G\). Note that \(G\) containing all but finitely many elements of \((a_n)\). Hence \(a_n \to 0\).

1.5 Definition ([1],[6])

Let \(f: X \to X\) be a continuous function. The iterates \(f^n\) form a group under composition. Then \((X, f)\) or \((f^n)_{n \in \mathbb{Z}^+}\) or \(z\) is called a discrete dynamical system.
Otherwise if \(n \in R\) or any interval of \(R\), then \((X, f)\) or \((f^n)_{n \in \mathbb{R}}\) is called a continuous dynamical system.

1.6 Example

Consider that the topological space \((R, \tau_u)\), and \(f: R \to R\) be a function define by \(f(x) = -3x\). The dynamical system define by \(f\) is \(\{(-3)^nx\}_{n \in \mathbb{Z}^+}\).

1.7 Definition [1]

Let \((X, \tau)\) be a topological space, and \(f: X \to X\) be a continuous function. For all \(x \in X\),
The orbit of \(x\) under \(f\) is the sequence \((x, f(x), f^2(x), \ldots, f^n(x), \ldots)\), and it is denoted by \(O(x)\).

1.8 Example

Consider that the topological space \((R, \tau_u)\), and \(f: R \to R\) be a function define by \(f(x) = \frac{1}{5}x\).
\(O(0) = (0, 0, 0, \ldots, 0, \ldots)\)
\(O(1) = (1, \frac{1}{5}, \frac{1}{125}, \frac{1}{625}, \ldots, (\frac{1}{5})^n, \ldots)\)
\(O(-1) = (-1, -\frac{1}{5}, -\frac{1}{125}, -\frac{1}{625}, \ldots, (\frac{1}{5})^n, \ldots)\)
\(O(2) = (2, \frac{2}{5}, \frac{2}{125}, \frac{2}{625}, \ldots, (\frac{1}{5})^n (2), \ldots)\)

Figure (1): Orbit of 1 and \(-1\) under \(f(x) = \frac{1}{5}x\)
1.9 Definition [1]

Let \((X, \tau)\) be a topological space, \(f : X \rightarrow X\) be a continuous function, and \(x_0\) be a fixed point of \(f\). Then \(x_0\) is called stable if for every open set \(U \subseteq X\) containing \(x_0\) there exists an open set \(V \subseteq U\) containing \(x_0\) such that, \(O(x) \subseteq U\ \forall\ x \in V\). Otherwise \(x_0\) is called unstable fixed point.

1.10 Example

Consider that the topological space \((R, \tau_0)\), and \(f : R \rightarrow R\) be, where
\[
\begin{align*}
\text{i) } f(x) &= \frac{1}{2}x. \quad \text{The dynamical system define by } f \text{ is } \left\{\left(\frac{1}{2}\right)^n x\right\}_{n \in \mathbb{Z}^+}, \text{ and } 0 \\
\text{is the fixed point of } f.
\end{align*}
\]
Let \(U = (a, b) \in \tau_0\), where \(0 \in U\).
Choose \(V = (-c, c) \in B_{\tau_0}\), where \(0 \in V \subseteq U\), \(c = \min \{|a|, |b|\}\). Note that \(O(x) \subseteq V \subseteq U\ \forall\ x \in V\). Hence 0 is stable.

\[
\begin{align*}
\text{ii) } f(x) &= 3x. \quad \text{The dynamical system define by } f \text{ is } \left\{(3)^n x\right\}_{n \in \mathbb{Z}^+} \text{ and } 0 \text{ is the fixed point of } f. \text{Let } U = (-1,1) \in \tau_0, \text{ note that } O(x) \not\subseteq U, x \in V, \forall \ V \in \tau_0, 0 \in V \subseteq U.
\end{align*}
\]
Hence 0 is unstable.
2. Main Results

2.1 Theorem

Let \((X, \tau)\) be a topological space, and \(B_\tau\) be a base for \(\tau\). The sequence \(\langle a_n \rangle_{n \in \mathbb{Z}^+}\) is convergent with respect to \(B_\tau\) iff it is convergent with respect to \(\tau\).

**Proof** (\(\Rightarrow\)): Let \(\langle a_n \rangle_{n \in \mathbb{Z}^+}\) be a convergent sequence with respect to \(B_\tau\), and \(x_0\) be a limit of \(\langle a_n \rangle_{n \in \mathbb{Z}^+}\). Let \(H\) be an open set containing \(x_0\). Then \(H = \bigcup_{a \in \lambda} W_a\) where \(W_a \in B_\tau\) \(\forall a \in \lambda\), then \(x_0 \in \bigcup_{a \in \lambda} W_a \Rightarrow x_0 \in W_{a_0}\) for some \(W_{a_0} \in B_\tau\). Since \(W_{a_0}\) containing all but finitely many elements of \(\langle a_n \rangle_{n \in \mathbb{Z}^+}\), and \(W_{a_0} \subseteq H\). Then \(H\) containing all but finitely many elements of \(\langle a_n \rangle_{n \in \mathbb{Z}^+}\). Hence \(\langle a_n \rangle_{n \in \mathbb{Z}^+}\) is convergent with respect to \(\tau\).

**Proof** (\(\Leftarrow\)): let \(\langle a_n \rangle_{n \in \mathbb{Z}^+}\) be a convergent sequence with respect to \(\tau\), and \(x_0\) be a limit of \(\langle a_n \rangle_{n \in \mathbb{Z}^+}\). Let \(H \in B_\tau\) containing \(x_0\). Since \(H\) is an open set, so \(H\) containing all but finitely many element of \(\langle a_n \rangle_{n \in \mathbb{Z}^+}\). Hence \(\langle a_n \rangle_{n \in \mathbb{Z}^+}\) is convergent with respect to \(B_\tau\).

2.2 Definition

Let \((X, \tau)\) be a topological space, \(f: (X, \tau) \to (X, \tau)\) be a continuous function, and \(x_0\) be a stable fixed point. We say that \(x_0\) is strongly stable if there exists an open set \(G\) containing \(x_0\) such that \(O(x) \to x_0\) \(\forall x \in G\). Otherwise we say that \(x_0\) is not strongly stable.

![Figure (5): Strongly stable fixed point](image)

2.3 Theorem

Let \((X, \tau)\) be a topological space, \(B_\tau\) is a base for \(\tau\), \(f: (X, \tau) \to (X, \tau)\) be a continuous function, and \(x_0\) be a fixed point of \(f\). If \(x_0\) is stable with respect to \(B_\tau\) then \(x_0\) is stable with respect to \(\tau\).

**Proof**: Let \(x_0\) be a fixed point of \(f\), and it is stable with respect to \(B_\tau\). Let \(U\) an open set containing \(x_0\). Since \(U = \bigcup_{a \in \lambda} W_a\), where \(W_a \in B_\tau\) \(\forall a \in \lambda\), then \(x_0 \in \bigcup_{a \in \lambda} W_a \Rightarrow x_0 \in W_{a_0}\) for some \(W_{a_0} \in B_\tau\). Hence \(x_0\) is stable with respect to \(\tau\).

2.4 Theorem

Let \((X, \tau)\) be a topological space, \(B_\tau\) is a base for \(\tau\), \(f: (X, \tau) \to (X, \tau)\) be a continuous function, and \(x_0\) be a fixed point of \(f\). If \(x_0\) is strongly stable with respect to \(B_\tau\) then \(x_0\) is strongly stable with respect to \(\tau\).

**Proof**: Let \(x_0\) be a fixed point of \(f\), and it is strongly stable with respect to \(B_\tau\). From Theorem(2.3) \(x_0\) is stable with respect to \(\tau\), suppose that \(\exists G \in \tau\), such that \(x_0 \in G\). Since \(G = \bigcup_{a \in \lambda} W_a\) where \(W_a \in B_\tau\) \(\forall a \in \lambda\), then \(x_0 \in \bigcup_{a \in \lambda} W_a \Rightarrow x_0 \in W_{a_0}\) for some \(W_{a_0} \in B_\tau\). Since \(O(x) \to x_0\) with respect to \(B_\tau\) \(\forall x \in \mathbb{R}\), \(\forall \tau\), \(U\) \(\subseteq U\). Hence \(x_0\) is strongly stable with respect to \(\tau\).

2.5 Example

Consider the topological space \((\mathbb{R}, \tau_u)\), and \(f: \mathbb{R} \to \mathbb{R}\) be a function define by

\[
i f(x) = \frac{1}{3} x
\]

The dynamical system define by \(f\) is

\[
\left\{ \left(\frac{1}{3} \right)^n x \right\}_{n \in \mathbb{Z}^+}
\]

and \(0\) is the fixed point of \(f\).
Let $U = (a, b) \in B_{r_u}$, where $0 \in U$. Choose

$V = (-c, c) \in B_{r_u}$, where $0 \in V \subseteq U$.

c = \min \{|a|, |b|\}. Note that

$O(x) \subseteq V \subseteq U \forall x \in V$. Then; $0$ is stable. Let

$G = (-\frac{1}{3}, \frac{1}{3}) \in B_{r_u}$ where $0 \in G$. Then

$O(x) \rightarrow 0 \forall x \in G$. Hence, $0$ is strongly stable.

Figure (6): Orbit of $9$ and $-9$ under $f(x) = \frac{1}{3}x$

The dynamical system define by $f$

is $\{x^{2^n}\}_{n \in \mathbb{Z}^+}$ and $-1, 0, 1$ are the fixed

points of $f$. $0$ is strongly stable and $1$ is

unstable. To show that $0$ is strongly stable and

$1$ is unstable. Let $U = (a, b) \in B_{r_u}$, where

$0 \in U$. Let $c = \min\{|a|, |b|\}$. If $c > 1$, $V =

(-1, 1)$ else if $c \leq 1$, $V = (-c, c)$, where

$0 \in V \subseteq U$. Note that $O(x) \subseteq V \subseteq U \forall x \in V$.

Hence $0$ is stable. Let $G = (-1, 1) \in B_{r_u}$ where

$0 \in G$. Then; $O(x) \rightarrow 0 \forall x \in G$. Hence, $0$ is

strongly stable. Now, let $U = \left(\frac{1}{2}, 2\right) \in B_{r_u}$.

Note that $O(x) \not\in U \forall V \in B_{r_u}, x \in V$,

$1 \in V \subseteq U$. Hence $1$ is unstable.

The dynamical system define by $f$

is $\{(-1)^n x\}_{n \in \mathbb{Z}^+}$ and $0$ is the fixed point of $f$.

Figure (7): Orbit of $\frac{1}{2} - \frac{5}{4}$ and $-\frac{5}{4}$ under $f(x) = x^2$
Note that $H$ does not contain any element of $O(x)$, so $\langle O(x) \rangle \to 0 \forall x \in G$. Hence 0 is not strongly stable.

2.6 Definition

Let $(X, \tau)$ be a topological space, $f: X \to X$ be a continuous function, and $x_0$ be a fixed point of $f$. Then we say that $x_0$ is c-stable if for every open set $U \subseteq X$ containing $x_0$ there exists an open set $V \subseteq U$ containing $x_0$ such that, $O(x) \subseteq \bar{U} \forall x \in V$.

Otherwise we say that $x_0$ is not c-stable fixed point.

2.7 Definition

Let $(X, \tau)$ be a topological space, $f: (X, \tau) \to (X, \tau)$ be a continuous function, and $x_0$ be a c-stable fixed point. We say that $x_0$ is strongly c-stable if there exists an open set $G$ containing $x_0$ such that $\langle O(x) \rangle \to x_0 \forall x \in G$. Otherwise we say that $x_0$ is not strongly c-stable.

2.8 Theorem

Let $(X, \tau)$ be a topological space, $B_\tau$ is a base for $\tau$, $f: (X, \tau) \to (X, \tau)$ be a continuous function and $x_0$ is a fixed point of $f$. If $x_0$ is stable then it is c-stable.

Proof: Since $U \subseteq \bar{U}$, then $O(x) \subseteq \bar{U} \forall x \in V$ by definition (1.10).

2.9 Theorem

Let $(X, \tau)$ be a topological space, $B_\tau$ is a base for $\tau$, $f: (X, \tau) \to (X, \tau)$ be a continuous
function and \( x_0 \) is a fixed point of \( f \). If \( x_0 \) is c-stable with respect to \( B_\tau \) then \( x_0 \) is c-stable with respect to \( \tau \).

**Proof:** Let \( x_0 \) be a fixed point of \( f \), and it is c-stable with respect to \( B_\tau \). Let \( U \) an open set containing \( x_0 \). Since \( U = \bigcup_{\alpha \in \Lambda} W_\alpha \), where \( W_\alpha \in B_\tau \forall \alpha \in \Lambda \), then

\[
x_0 \in U_{\alpha} \Rightarrow x_0 \in W_{\alpha_0} \text{ for some } W_{\alpha_0} \in B_\tau \text{. Since is stable with respect to } B_\tau \text{ then } \exists \, V \in B_\tau \text{ such that } x_0 \in V \subseteq W_{\alpha_0} \text{ and } O(x) \subseteq W_{\alpha_0} \forall x \in V \text{. Note that } V \in \tau \text{ and } W_{\alpha_0} \subseteq U \subseteq \bar{U}, \text{ so } O(x) \subseteq W_{\alpha_0} \subseteq \bar{U} \forall x \in V \text{. Hence } x_0 \text{ is c-stable with respect to } \tau \text{.}
\]

**2.10 Theorem**

If \( x_0 \) is strongly c-stable with respect to \( B_\tau \) then \( x_0 \) is strongly c-stable with respect to \( \tau \).

**Proof:** Let \( x_0 \) be a fixed point of \( f \), and it is strongly c-stable with respect to \( B_\tau \). From Theorem(2.9) \( x_0 \) is c-stable with respect to \( \tau \), suppose that \( \exists G \in \tau \) such that \( x_0 \in G \). Since \( G = \bigcup_{\alpha \in \Lambda} W_\alpha \text{ where } W_\alpha \in B_\tau \text{ }\forall \alpha \in \Lambda \), then \( x_0 \in \bigcup_{\alpha \in \Lambda} W_\alpha \Rightarrow x_0 \in W_{\alpha_0} \) for some \( W_{\alpha_0} \in B_\tau \). Since \( O(x) \rightarrow x_0 \) with respect to \( B_\tau \forall x \in W_{\alpha_0} \subseteq U \). Hence \( x_0 \) is strongly c-stable with respect to \( \tau \).

**2.11 Example**

Consider the topological space \( (R, \tau_u) \), and \( f: R \rightarrow R \) be a function define by

\[
i) \quad f(x) = \frac{1}{4}x
\]

The dynamical system define by \( f \) is \( \left\{\left(\frac{1}{4^n}x\right)\right\}_{n \in \mathbb{Z}^+} \text{, and } 0 \text{ is the fixed point of } f \text{.}
\]

Let \( U = (a, b) \in B_{\tau_u} \) where \( 0 < a \). Choose 
\[
\begin{align*}
\text{Let } U &= (a, b) \in B_{\tau_u}, \text{ where } 0 < a. \text{ Choose } V &= (-c, c) \in B_{\tau_u}, \text{ where } 0 \leq V \subseteq U, \\
&c = \min \{|a|, |b|\}. \text{ Note that } O(x) \subseteq V \subseteq \bar{U} \forall x \in V. \text{ Then } 0 \text{ is } c\text{-stable.}
\end{align*}
\]

Let \( G = \left(-\frac{1}{4^n}, \frac{1}{4^n}\right) \in B_{\tau_u} \text{ where } 0 \in G \).

\[
\text{Then } \langle O(x) \rangle \rightarrow 0 \forall x \in G.
\]

Hence 0 is strongly c-stable.

\[
i) \quad f(x) = -2x
\]

The dynamical system define by \( f \) is \( \left\{\left(-2^n x\right)\right\}_{n \in \mathbb{Z}^+} \text{, and } 0 \text{ is the fixed point of } f \text{.}
\]

Let \( U = (-1,1) \in B_{\tau_u} \). Note that \( O(x) \subset V \neq U \forall V \in B_{\tau_u} \), \( x \in V, 0 \leq V \subseteq U \).

Hence 0 is not c-stable fixed point.

\[
ii) \quad f(x) = 1 - x
\]

The dynamical system define by \( f \) is \( \left\{\begin{align*}
1 - x & \text{ if } n \text{ odd} \\
x & \text{ if } n \text{ even} \end{align*}\right\}_{n \in \mathbb{Z}^+} \text{, and } \frac{1}{2} \text{ is the fixed point of } f \text{.}
\]

Let \( U = (a, b) \in B_{\tau_u} \) where \( \frac{1}{2} \in U \). Let \( c = \frac{1}{2} - a \), and \( d = |b - \frac{1}{2}| \).

If \( c \leq d \), choose \( V = \left(\frac{1}{2} - r, \frac{1}{2} + r\right) \in B_{\tau_u} \text{ where } r = \frac{|1 - c|}{2} \text{, if } c > d \text{, choose } V = \left(\frac{1}{2} - l, \frac{1}{2} + l\right) \in B_{\tau_u} \text{ where } l = \frac{|1 - d|}{2} \text{, where } \frac{1}{2} \in V \subseteq U \).

Note that \( O(x) \subseteq V \subseteq \bar{U} \forall x \in V \). Then \( \frac{1}{2} \) is c-stable.

Let \( G = (a, b) \in B_{\tau_u} \) where \( \frac{1}{2} \in U \). Let \( x \in G \), and \( H = (-x, x) \in B_{\tau_u} \text{ where } \frac{1}{2} \in H \).

Note that \( H \) does not contain any element of \( O(x) \), so \( \langle O(x) \rangle \nrightarrow \frac{1}{2} \forall x \in G \).

Hence \( \frac{1}{2} \) is not strongly c-stable.
Figure (11): Orbit of 1 and $-1$ under $f(x) = \frac{1}{4}x$

Figure (12): Orbit of $\frac{1}{2}$ under $f(x) = -2x$

Figure (13): Orbit of 1, 2, and 3 under $f(x) = 1 - x$
References


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